Patterns and Notions

'The Joy of Qualitative Geometry Study'

Wayne (Skip) Trantow
Omnigarten.org

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Prologue

One of the great joys of life is observing 'things of nature' with as few assumptions as possible. When we suspend assumption, new perspectives become possible. When we focus our observation on fundamental Geometric form it is indeed fertile ground for study, for here we are at a place where the physical and the universal are close. We recognize this strategy in the work of the classical Greek philosophers who forged insight through the study of regular polyhedral solids. We can intuit, as they did, that fundamental Geometric form reveals intrinsic universal principles.

In contrast to the classical approach to Geometry study, which is characteristically static and quantitative, we undertake an informal qualitative study of Geometric form as it grows. From observations made in progressive buildout of node-based Geometric models we learn a vocabulary of Nature's 'patterning notions'[1], a vocabulary that creates a conceptual fabric tangible enough for our reasoning to work with. When we think in terms of notions, we can sense 'how' Nature expresses itself physically.

[1] *Throughout this essay the term 'patterning notions' refers to a consistent underlying dynamic that is present in the geometric organization of patterns as they grow. While the term 'design principle' could be used, describing our observations as 'notions' relates them more to intuition, whereas the term 'principles' seems to correspond more to analytical reasoning. Intuitive sensing is more subtle - a necessity on this journey.*

Two Journeys

Geometric pattern is a language of its own. Like music it conveys meaning beyond what can be described by words, symbols, or number. This language guides our awareness into the realm of organizing notions behind Nature's form making.

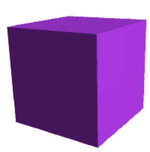
Models are useful and indeed necessary to learn the language of Geometry. Furthermore, it is crucial to work with three-dimensional (3D) models, as opposed to two-dimensional diagrams, for after all, Nature's forms are three-dimensional. In the process of 3D modeling can study structures from different viewpoints, something not easily done with drawings or in the quantitative analyses found in your typical geometry textbook. 3D models held in hand generously reveal the fundamental notions that guide Nature's physicalizing.

Let's begin a journey into Geometric form study by metaphorically entering Plato's Academy. We start here as homage to human history, for this is a seminal chapter in the story of humanity seeking the bedrock principles behind life and physical manifestation. We start here also to delve into a perspective that was cemented by the classical Greek philosophers and instilled into the world view that we share today.

Quantitative Study

As we enter the Academy, we notice the beautiful furniture - the iconic Platonic Solids.

The tetrahedron is the Platonic solid with three triangular faces arranged around every vertex. Plato identified this polyhedron with the shape of fire atoms.



The cube is the Platonic solid with three square faces arranged around every vertex. Plato identified this polyhedron with the shape of earth atoms.

The octahedron is the Platonic solid with four triangular faces arranged around every vertex. Plato identified this polyhedron with the shape of air atoms.

The icosahedron is the Platonic solid with five triangular faces arranged around every vertex. Plato identified this polyhedron with the shape of water atoms.

The dodecahedron is the Platonic solid with three pentagonal faces arranged around every vertex. Plato identified this polyhedron with the shape of cosmos atoms.

Table 1: Platonic Solids

What characterizes the Platonic Solids as a geometrical set is that the face of each solid is a regular polygon of the same size and shape. We instinctively see them as standalone 'objects' - beautiful rocks if you will.

What characterizes the classical Greek approach to form study is that it is '**rooted in number'**, i.e., these solids conform to the philosophers' intent to represent primary numbers (e.g., 3,4,5) as physical form. This approach inherently begets counting, measuring and classification, demonstrated in Table 2 by the various descriptors of the Platonic Solids:

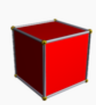
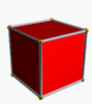
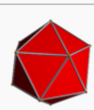
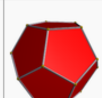
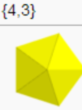
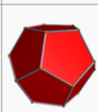
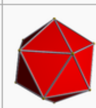
Index	Name	Picture	Dual name	Dual picture	Wythoff symbol	Vertex figure and Schläfli symbol	Symmetry group	U#	K#	V	E	F	Faces by type
1	Tetrahedron		Tetrahedron		3|2 3	 {3,3}	T_d	U01	K06	4	6	4	4{3}
2	Octahedron		Hexahedron		4|2 3	 {3,4}	O_h	U05	K10	6	12	8	8{3}
3	Hexahedron (Cube)		Octahedron		3|2 4	 {4,3}	O_h	U06	K11	8	12	6	6{4}
4	Icosahedron		Dodecahedron		5|2 3	 {3,5}	I_h	U22	K27	12	30	20	20{3}
5	Dodecahedron		Icosahedron		3|2 5	 {5,3}	I_h	U23	K28	20	30	12	12{5}

Table 2: Characterization of the Platonic Solids

It also led to a system of transformations (e.g., expansion, stellation, truncation, etc.) to derive duals and other forms, and a nomenclature to identify them, as demonstrated in Table 3 that describes the thirteen Archimedean Solids.

Name/ (alternative name)	Schläfli Coxeter	Transparent	Solid	Net	Vertex conf./fig.	Faces		Edges	Vert.	Volume (unit edges)	Point group	Sphericity
truncated tetrahedron	t{3,3}				3.6.6	8	4 triangles 4 hexagons	18	12	2.710 576	T_d	0.775 4132
cuboctahedron (rhombitetratetrahedron)	r{4,3} or rr{3,3}				3.4.3.4	14	8 triangles 6 squares	24	12	2.357 023	O_h	0.904 9973
truncated cube	t{4,3}				3.8.8	14	8 triangles 6 octagons	36	24	13.599 663	O_h	0.849 4937
truncated octahedron (truncated tetratetrahedron)	t{3,4} or tr{3,3}				4.6.6	14	6 squares 8 hexagons	36	24	11.313 709	O_h	0.909 9178
rhombicuboctahedron (small rhombicuboctahedron)	rr{4,3}				3.4.4.4	26	8 triangles 18 squares	48	24	8.714 045	O_h	0.954 0796
truncated cuboctahedron (great rhombicuboctahedron)	tr{4,3}				4.6.8	26	12 squares 8 hexagons 6 octagons	72	48	41.798 990	O_h	0.943 1657
snub cube (snub cuboctahedron)	sr{4,3}				3.3.3.3.4	38	32 triangles 6 squares	60	24	7.889 295	O	0.965 1814
icosidodecahedron	r{5,3}				3.5.3.5	32	20 triangles 12 pentagons	60	30	13.835 526	I_h	0.951 0243
truncated dodecahedron	t{5,3}				3.10.10	32	20 triangles 12 decagons	90	60	85.039 665	I_h	0.926 0125
truncated icosahedron	t{3,5}				5.6.6	32	12 pentagons 20 hexagons	90	60	55.287 731	I_h	0.966 6219
rhombicosidodecahedron (small rhombicosidodecahedron)	rr{5,3}				3.4.5.4	62	20 triangles 30 squares 12 pentagons	120	60	41.615 324	I_h	0.979 2370
truncated icosidodecahedron (great rhombicosidodecahedron)	tr{5,3}				4.6.10	62	30 squares 20 hexagons 12 decagons	180	120	206.803 399	I_h	0.970 3127
snub dodecahedron (snub icosidodecahedron)	sr{5,3}				3.3.3.3.5	92	80 triangles 12 pentagons	150	60	37.616 650	I	0.982 0114

Table 3: Transformations of the Platonic Solids (The Archimedean Solids)

The classical philosophers sought metaphysical meaning by associating the Platonic Solids to physical elements, e.g., fire, earth, air, water, and the ether. While we no longer make such associations, we still employ the same perspective in our understanding of the world. Consider that the nomenclature of modern sciences, e.g., chemistry, metallurgy, crystallography, etc., are based upon the nomenclature of the Platonic Solids. It is thus important to recognize that the classical Greeks conferred dominance to a certain modality of comprehending physical form - a static, '**particle-mode'**, perspective. This is a valid and extremely useful modality of comprehension. Through it our physical world has been well studied and quantified to great benefit of mankind, but this inherited classical perspective has also constrained our understanding of the physical. I argue that it has obscured a larger significance.

Qualitative Study

Let us exit the Academy and re-explore fundamental geometric form in a different way; by simply observing, with as few assumptions as possible, geometric pattern as it emerges in growth. We embark on a **qualitative** analysis where we study geometric patterns using a skeletal 3D modeling tool. There is very little counting, measuring or mathematics; we work with relative proportion as opposed to absolute measurement. We root our exploration not in number but in a **thought experiment**. It goes like this:

Imagine that nothing exists except pure space within which a single object is created - the 'first' point. The first point is of arbitrary size (dimensionless) and we likely imagine a spherical object. The key characteristic here is that it is distinct from and bounded within the containing space which is unbounded and not containable.

We extrapolate that where there is room for the first object in unbounded space there is room for another to exist. The only possible next action in this context is that a second object is created in the exact likeness of the first object because that is all that is known. (*This is a core postulate that implies uniform spatial equilibrium in the growth of points.)* The two objects, however, must be distinct else they would occupy the same space and not be distinguishable. As such, the two objects find themselves simultaneously connected and separated in an equilibrium of attraction and repulsion. We might say that the forces of gravity (centripetal) and radiance (centrifugal) are at play here. The two objects and the equilibrium (i.e., state of balance) of forces between them define a line in space - the first dimension. This line is the base dimension from which all other lines will be understood – it is called 'unit length'.

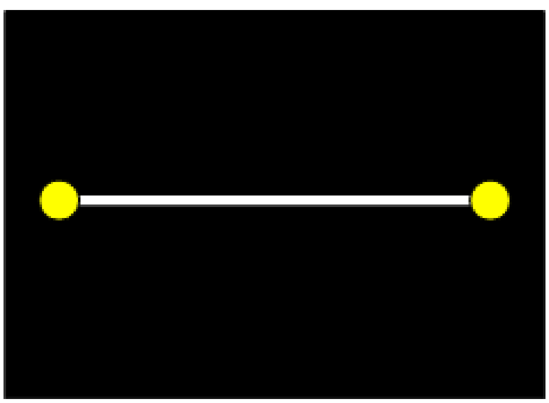


Figure A1: Line Segment of 'unit-length' formed by two points in space.

In the historical context, a line of unit-length is the basis of the 'Vesica Pisces' diagram shown in Figure A2.

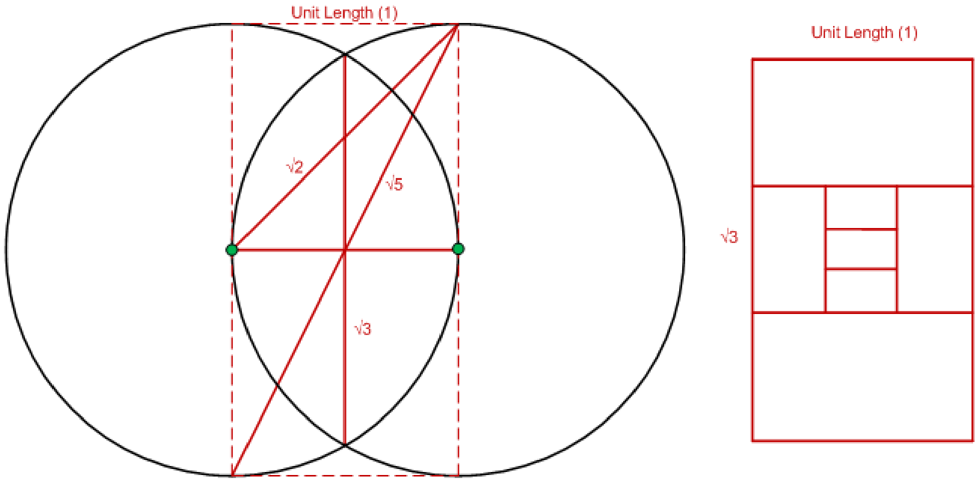


Figure A2: Vesica Pisces and an Example of Derived Rectangles

In the Vesica Pisces a circle of unit-length radius surrounds both the first and second points. The circle represents the identity of the object in terms of its force 'radiating' out omnidirectionally in two dimensions (i.e., its radius), as well as its force of attraction. Ancient geometers derived several important proportions from this pattern: √2, √3 and √5 relative to unit-length (1). These proportions were used to create sacred rectangles which served as aesthetic design frameworks in Egyptian, Greek, and other ancient art and architecture. An arbitrary rectangle composition based on √3 is shown on the right. It represents a harmonized set of proportions to guide aesthetic design.

As we continue the creation sequence and add more and more points, each one held in place at unit-length from adjacent points in a plane, we arrive at the 'Flower of Life' pattern shown in Figure A3. This beautifully interconnected pattern shows how an infinite cascade of point replications can fill our two-dimensional space.

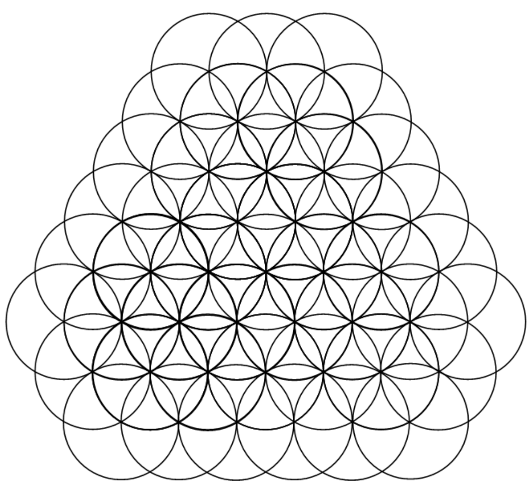


Figure A3: Flower of Life

But simply extrapolating the creation sequence in two dimensions, as the Flower of Life does, is a digression from the journey. In reality, objects (i.e., points) will replicate and come into spatial equilibrium in three dimensions, not just two. While our third replicated point would arrange itself to define a plane that includes the first two points, our fourth point would come into unit-length equilibrium at a right angle off of that established plane forming the Tetrahedron as shown in Figure A4. The Tetrahedron is the most basic 3D form. Its four points are in perfect spatial equilibrium at unit-length distance from each other.

Figure A4: Tetrahedron – Four Points in Spatial Equilibrium

Thus far we can easily visualize this pattern of points in our imagination but to study how this equilibrium of four identical points, the Tetrahedron, can grow in space we must augment our thought experiment with 3D skeletal geometric modeling tools. As we proceed, we will refer to these forms by classical Platonic names, but we will not necessarily view them from a classical 'solids' perspective. In fact, we must be very careful to not taint our exploratory perspective with the classical perspective.

It is also important to remember that, in node-and-strut modeling, the ***node*** *patterns and spatial relationships are the most important. The struts are the physical scaffolding that pull the nodes into patterns of interest per our requirement of equality of force / unit-length distance. So, while both nodes and struts make for beautiful imagery, the node patterns are more germane to our thinking about spatial form. Struts, per se, are like 'visual force traces' that enhance pattern viewing and understanding.*

Continuing, we can add a new point off of each of the faces of the Tetrahedron and still maintain the unit-length equilibrium of forces. This results in the Stellated Tetrahedron form of Figure A5.

Figure A5: Stellated Tetrahedron

But after this we can no longer add points off the faces and maintain coherent unit-length equilibrium of points. This is demonstrated in Figure A6. While a new point, as represented by the Tetrahedron in gray, can equalize off the face of the yellow Tetrahedron, it cannot equalize with the point of the Tetrahedron in green. The node distance, as shown by the red line, is greater than unit-length. We appear to have hit a dead-end.

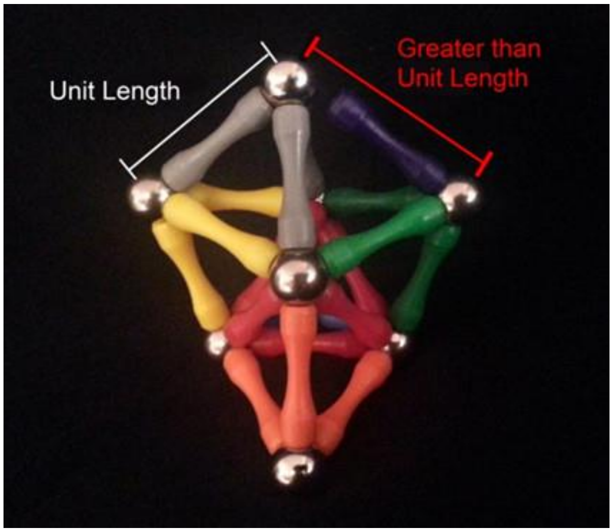


Figure A6: Tetrahedral Buildout Dead-end

But there is more to notice about the equilibrium of four points of our initial Tetrahedron. We observe that each node is perpendicular to the mid-point of the opposing Triangle plane. We also notice that each edge, represented by the white struts in Figure A7, is perpendicular (or orthogonal) to the edge opposite it.

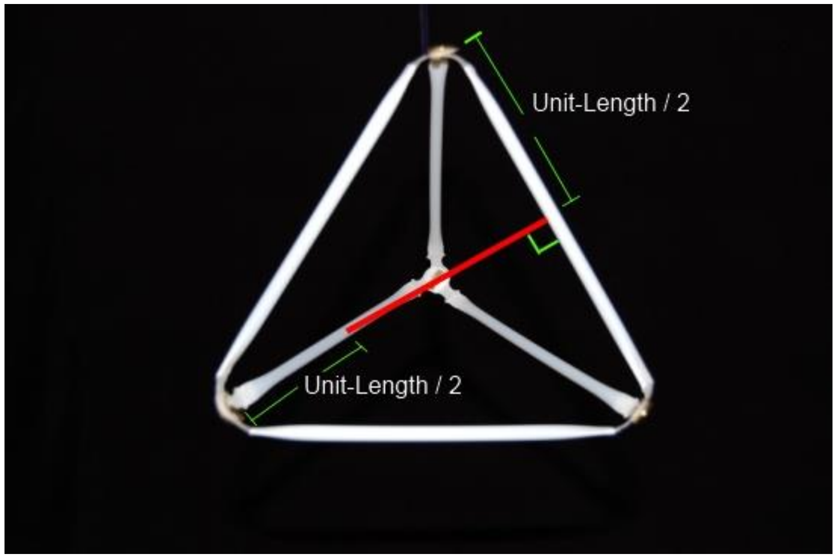


Figure A7: Opposing Edges of Tetrahedron are at 90º Rotation

If we were to connect the mid-points of these two lines as shown by the red line in Figure A7 and look straight down the red line the pattern of points would appear as a perpendicular cross as shown in Figure A8.

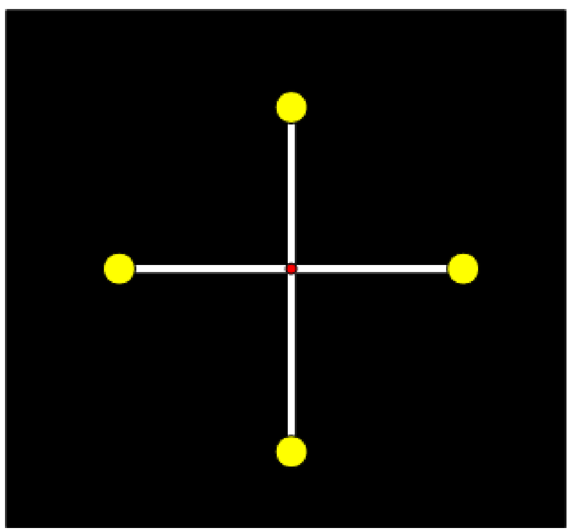


Figure A8: View of Tetrahedron nodes looking from mid-point of one line to its opposite

This pattern is the first signature of our initial assumption, that all points have the same power of attraction and radiance and thus coalesce into an equal distance of separation.

(*If we were to assume that any arbitrary point can have any arbitrary size/power, then coherency of spatial patterning would go 'out the window'.)*

There are three pairs of opposing struts in the Tetrahedron therefore three orthogonal axes (and planes) are present as shown in red in Figure A10.

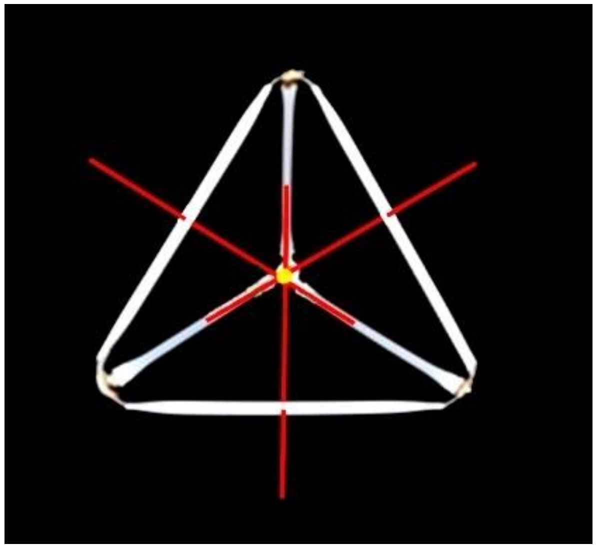


Figure A10: Orthogonal Axes of the Tetrahedron

In this we can we appreciate how our 3D Cartesian axes system is intrinsic to the uniform Tetrahedron equilibrium, but even more importantly, this spatial relationship foreshadows the emergence of the ‘Square’ and the ability of Nature to fill space. Let’s continue to add more points but now in a different manner. Instead of adding points that equalize relative to form ‘faces’ we add points to form lines.

In Figure B1 two Tetrahedrons are oriented such that an edge (strut) of each form is linearly aligned. This is highlighted by the white line at the top where three points form a straight line of two unit-lengths. Linear orientation of points[2] is a major leap from our initial building heuristic of simply equalizing a new point off an existing face, which we have observed is a dead-end for form growth. When incorporating linear orienting in our building approach, we find that two Tetrahedrons create a Pyramid form, and here (to thunderous applause) we have the grand premier emergence of the ‘Square’ (highlighted in red in Figure B1).

[2] *In mathematics two points define a conceptual line. In 3D Geometry, however, we find that a functional line is really determined by three points, for, there is one, and only one, alignment of three points in 3D space that creates a straight line.*

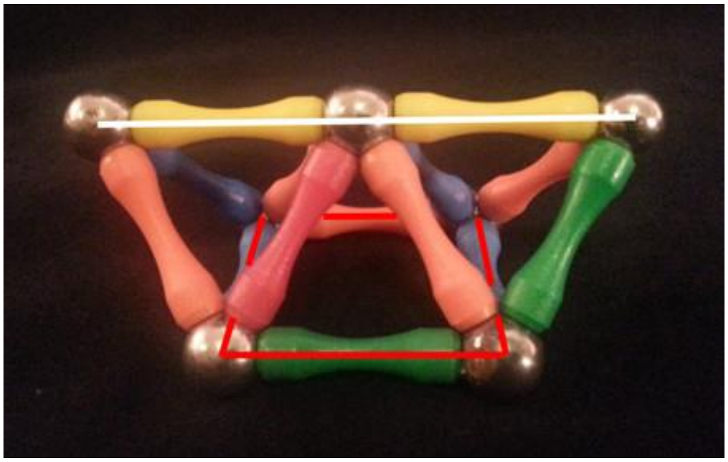


Figure B1: Linearly Arranged Tetrahedrons Creating Pyramid (Octahedron)

This exciting observation bears restating: At first blush we all probably think of the Tetrahedron as a pointy triangular object. But intrinsic to the unit-length equilibrium of four points that creates the Tetrahedron are three pairs of equal and orthogonal opposing edges (like the pair highlighted in white in Figure B2). This crisscross of point pairs is the predecessor of the Square pattern that emerges in subsequent buildout. In retrospect we can now recognize that the Square pattern is latent (hidden) in the Cross, and the Cross pattern is latent (hidden) in the premise of uniform separation of the four points of the Tetrahedron.

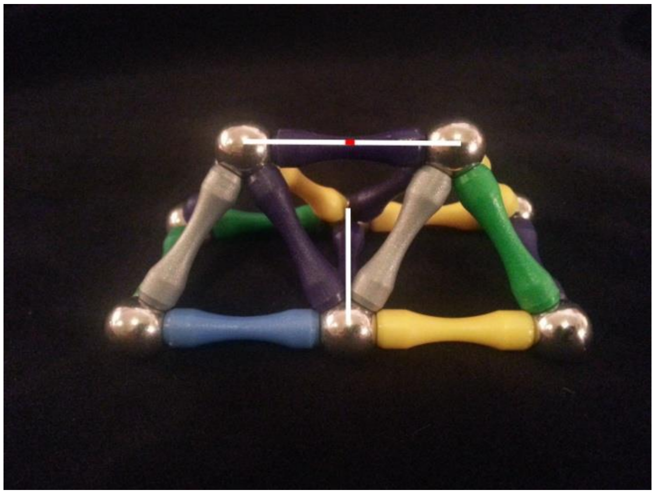


Figure B2: The Orthogonality of the Tetrahedron

Adding two more Tetrahedrons, for a total of four, results in a fully formed Octahedron shown in Figure B3. On the left the upper Tetrahedrons are aligned in parallel to the lower set. On the right the upper Tetrahedrons are aligned orthogonal to the lower set.

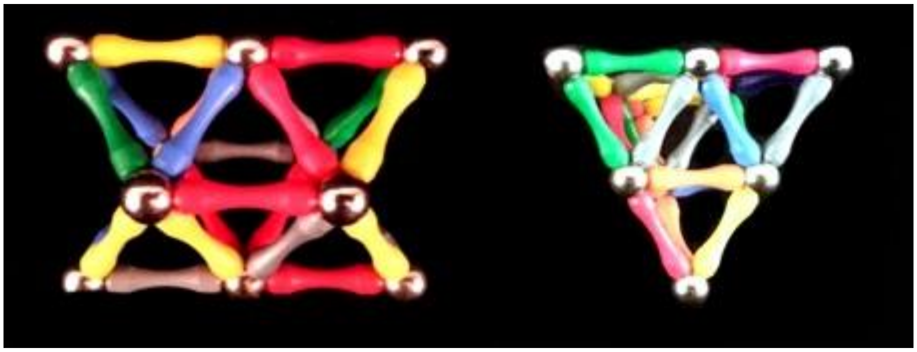


Figure B3: Four Tetrahedrons forming an Octahedron

Figure B4 shows eight Tetrahedron fully enclosed to form the Octahedron at center.

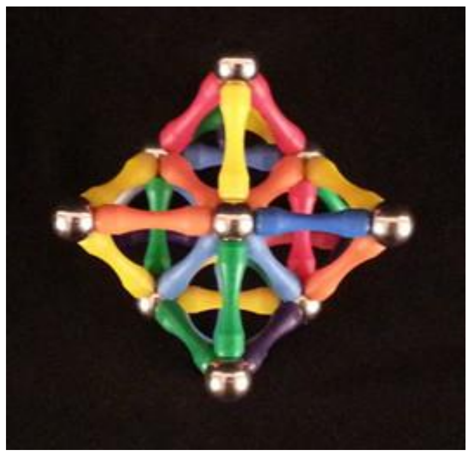


Figure B4: Eight Tetrahedrons forming the Octahedron (Stella Octangula)

In the classical perspective, this form is recognized as a standalone object called the Stellated Octahedron ('Stella Octangula'). However, from the perspective of our 'patterns-in-growth' study, it is a form that teaches us how a cluster of eight Tetrahedrons in 3D space creates the pattern of the Octahedron, hence the Square. (*And also, the eight outer nodes of the Tetrahedrons form a √2 Cube. More on that later.)*

An important observation here is that, in 3D form growth, unit-length Squares are created from unit-length Triangles by a force balancing dynamic that results in a linear alignment of three points, at unit-length separation. When we extrapolate to point forces balancing in three dimensions, the result is the orthogonal arrangement of unit lines visualized (in a perspective view) in Figure B5. Considering our first point at center, six other points (of equal force) equalize by **extent** (distance) from the center point and by extent from their four closet neighbors in an equilibrium of **direction**. As more points are added this equilibrium produces a cubic lattice.

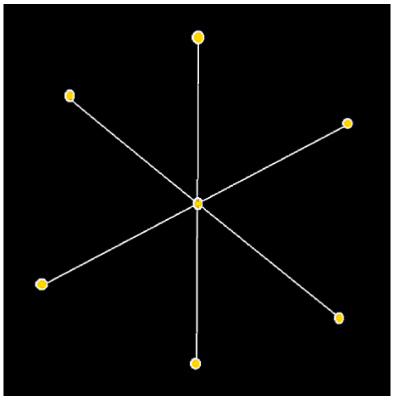


Figure B5: Equilibrium of Extent and Direction

Triangle and Square are thus, not two independent forms, but simply different '**aspects**' of point equilibrium. Triangle and Square are Nature's simple and elegant principle behind a vast diversity of complex physical form. In this we recognize a fundamental and powerful patterning notion – the notion of '**Duality**'[3]. We will continue see this notion at play in our buildout of points.

Triangle and Square Buildouts

Let's continue exploring using the set of the simplest compositions of 3D form, comprised solely of unit-length Triangles and/or Squares, as a starting base. Referred to as the 'Base Forms' they are shown in Figure C1. We refer to them by classical Platonic names but, again, we are being careful to not view them from a classical 'solids' perspective.

[3] *The notion of Duality here is not to be confused with the notion of Duals in classical solid geometry whereby a dual solid is created out of an existing solid by transforming its vertices into faces and faces into vertices.*

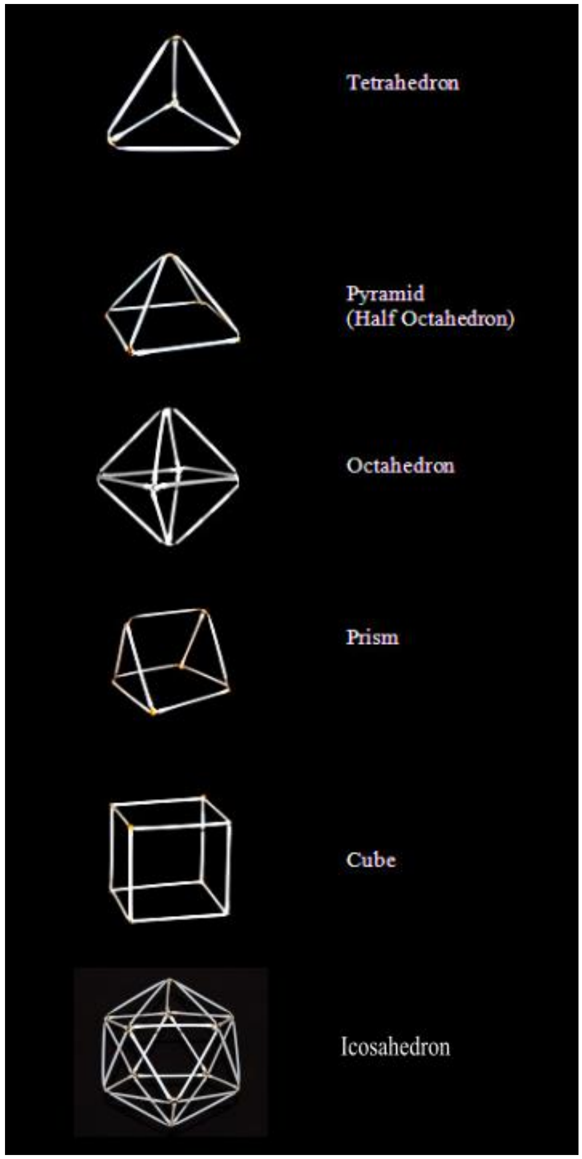


Figure C1: Base Forms of Triangle and/or Square

Of the Base Forms there are ‘heterogeneous’ forms, the Octahedron (and Pyramid) and Prism, composed of both Triangle AND Square patterns, and ‘homogeneous’ forms, the Tetrahedron, Cube and Icosahedron, composed of only Triangle OR Square patterns.

Each tells a story in growth:

Tetrahedron Buildout

As we saw above an omnidirectional buildout of Tetrahedrons off its faces quickly reaches a dead-end. But, as we will see later, Tetrahedrons can be linearly aligned off its faces to create an infinitely long linear tube comprised solely of Triangle faces. When the Tetrahedron is aligned linearly along its edges it derives Octahedron patterns. These patterns complement each other to allow for infinite growth as a Tetrahedral-Octahedral Lattice[4].

Cube Buildout

It is easy to imagine omnidirectional buildouts of Cubes. It results in either a larger cube or variations of rectangular boxes. We witness cubic buildout every day in the myriad shapes of our buildings, living spaces, cabinets, boxes, etc. Variation is possible in the sense of branches that travel out and along any of three orthogonal axes of the Cube faces, or by interfacing with the square faces of a Prism to fork it. Such branches can intersect with other branches to create infinite variations. Compared to the heterogeneous Triangle-Square Lattice networks (i.e., the Tetrahedral-Octahedral Lattice), a <u>unit-length</u> Cubic Lattice is <u>not</u> a stable structure, as only 'triangulated' node arrangements are truly stable. However, as will be discussed later, √2 Cubes are inherent to the Tetrahedral-Octahedral Lattice, and thus are stable. A √2 Cubic Lattice is an aspect of the Tetrahedral-Octahedral Lattice.

Prism Buildout

The Prism is a heterogeneous form that can multiply coherently around one axis of the square facet to form a hexagonal prism and then can grow infinitely long along that same axis. Multiple hexagonal prisms can be stacked in an infinite honey-comb pattern, but the Prism form by itself has diminished power for variation in growth. We will not explore Prism buildout in this essay.

Octahedron Buildout

As seen above the Octahedron is a child of Tetrahedrons. Perspective is, once again, important as the Octahedron form can be recognized in multiple ways. In the classical 'solids' view it is seen as the formation of eight equilateral unit-length triangle faces. In the 'skeletal' view we can see it as one square and eight triangles (or two mated Pyramids with a square bottom and four triangular sides). Another perspective is to see it as **three Squares** that intersect as orthogonal 3D planes as shown in Figure C2 where one square is White, one Red and one is Red-White Dashes.

[4] This lattice pattern was used by Alexander Graham Bell in his large truss-works, i.e., spaceframes. It was called the 'Octet Truss' by Buckminster Fuller.

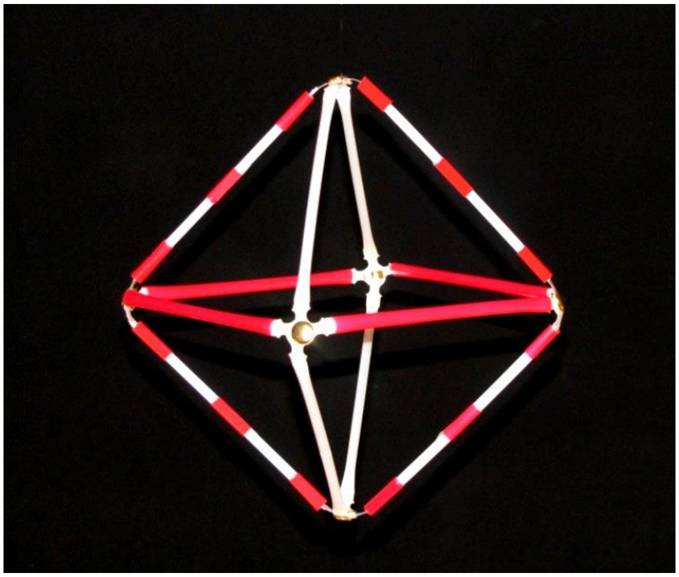


Figure C2: Octahedron as Three Orthogonal Squares

Let’s explore an Octahedron buildout relative to its Square plane and its Triangle plane. In Figure C3 is a buildout on the plane of its Triangle faces. As can be seen the growth results in a larger Tetrahedron. Each face of the larger Tetrahedron is a plane of Triangles (the ‘Flower of Life’ pattern).

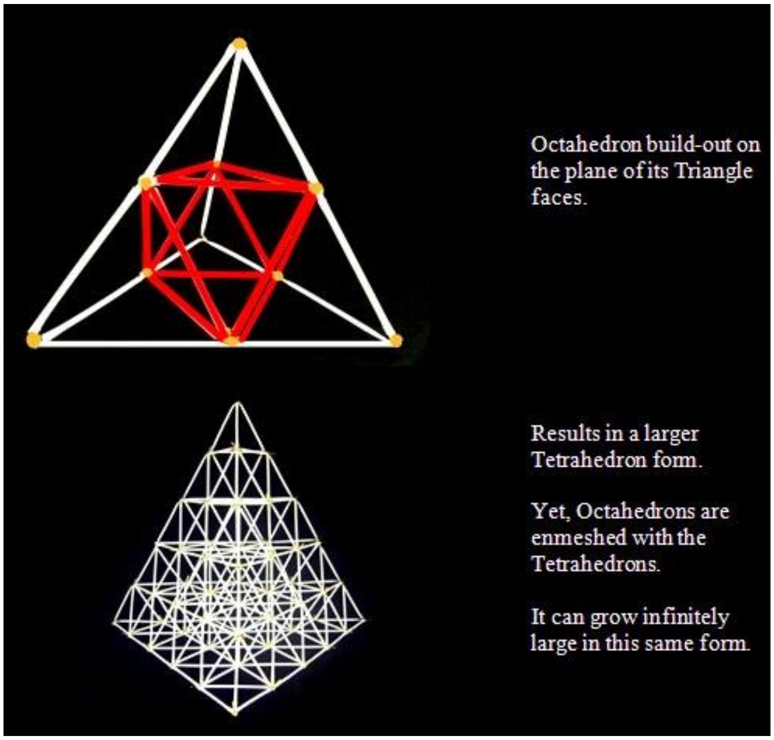


Figure C3: Tetrahedron Comprised of Octahedrons and Tetrahedrons

If we build out along the plane of its Square face it grows into a larger Octahedron as shown in Figure C4.

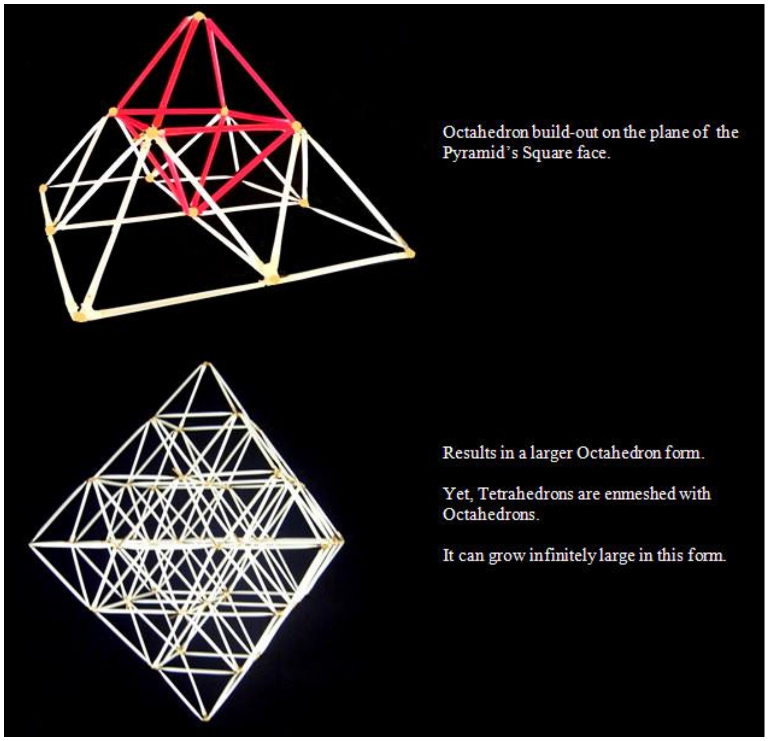


Figure C4: Octahedron comprised of Octahedrons and Tetrahedrons

These two buildouts reveal the power of the Triangle and Square to grow and fill space as intersecting planes - a 'mineral-like' growth. It grows as an Octahedron when building from the plane of the Squares and grows as a Tetrahedron when building from the plane of its Triangles. In either case, it is the same form, referred to in classical geometry study as the Tetrahedral-Octahedral Lattice. As we will see later, in Figure C9, it is just a matter of which sub-form (or aspect) you select to view.

In the Tetrahedral-Octahedral Lattice there are eight planes of Triangles (per Figure P1) intersecting with three planes of Squares (per Figure P2). Parallel planes of Squares intersect orthogonally like the Squares of Figure C2. Parallel planes of Triangles follow the eight faces of any given Octahedron in the lattice. The lattice can grow infinitely large.

Vector Equilibrium

A study of Octahedrons would not be complete without an exploration of the Cuboctahedron, also called the 'Vector Equilibrium' (VE) by Buckminster Fuller. It is shown in Figure C5.

Figure C5: Vector Equilibrium – Cuboctahedron

As a standalone form the VE is symmetrical across reflections of four Hexagon planes, formed by six unit-length Triangles as shown in Figure C6. You can see the four Hexagon planes of the VE in Figure C5. They are highlighted in red, yellow, blue and green at their perimeters. The Hexagon planes intersect at their center.

In classical characterization the VE is described as having eight Triangles and six Squares for a total of fourteen outer faces. Each edge of the six Squares of the VE is one strut of the perimeter edge of the Hexagon planes. It can be decomposed into eight Tetrahedrons and six Pyramids.

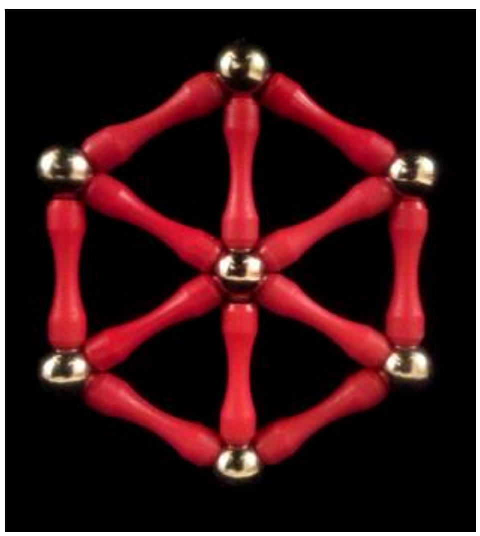


Figure C6: One of four Hexagon Planes of the VE

But on further examination we see that the VE is just a sub-form within the Tetrahedron/Octahedron Lattice pattern shown in Figures C3 and C4. The VE is latent in the Lattice pattern and emerges when the Lattice has an edge of at least three unit-lengths as shown in Figure C7. The form on the right shows the VE sub-form highlighted with yellow lines. A portion of the VE is on the other side of the red Hexagon plane. The vector lengths are one unit-length. However, if one were to continue building out the Pyramid, at a certain size (i.e., node count) a larger VE form would emerge with vector lengths greater than one unit-length.

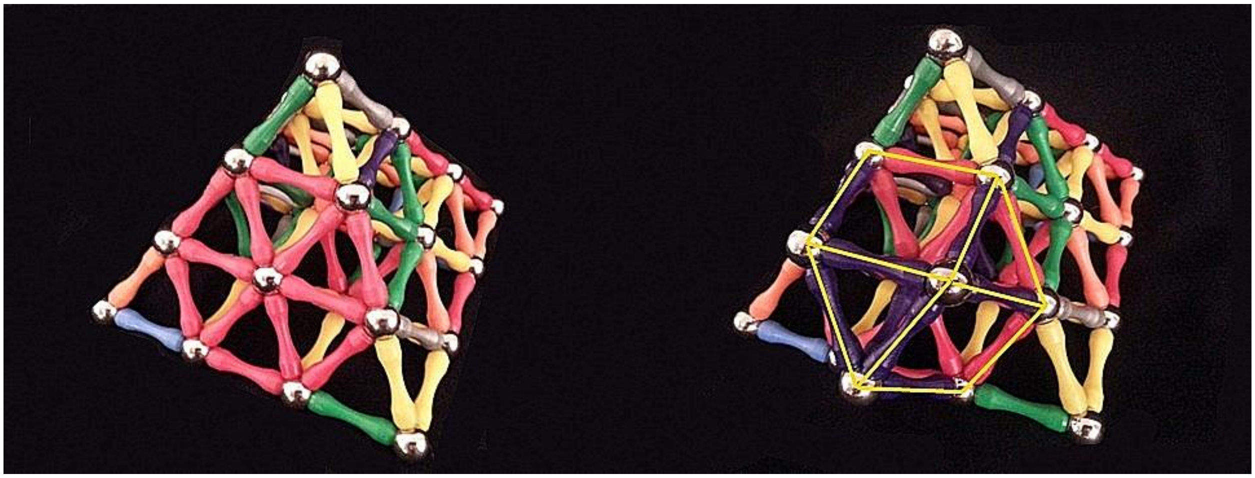


Figure C7: VE Inside the Pyramid

Whereas the Pyramid form does not have a center of balance, the VE as a stand-alone form does. Its center node is unit-length distance to all nodes in the outer layer and all nodes in the outer layer are unit-length Triangles and Squares. Indeed, it is because <u>all</u> the struts are unit-length that it is called the 'Vector Equilibrium', and in this characteristic, we may recognize the VE as the second manifestation of point equilibrium after the Tetrahedron.

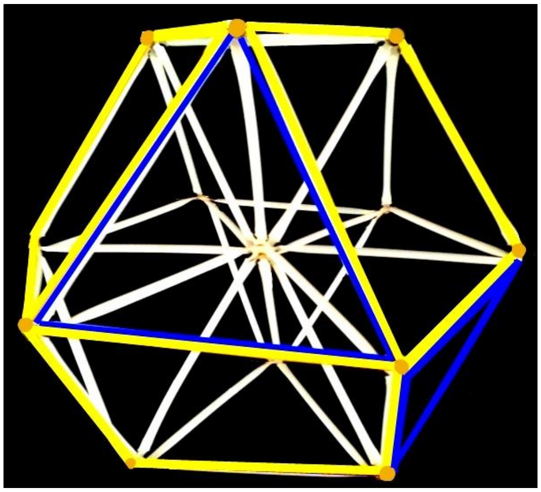


Figure C8: Unit-length Triangles and Squares and Interior Struts of the VE

Recall that in the Tetrahedron we spied the precursor of the Square in the orthogonally opposing edges. A linear aligning of Tetrahedrons created the Octahedron and the emergence of the Square. Further growth in planar orientation revealed the potential for infinitely large Lattices comprised of Tetrahedrons and Octahedrons. Fundamental to Lattice growth is the repetition of Triangle and Square patterns. The VE is the simplest distillation of the harmony of Triangle and Square in three dimensions. It is the very essence of the Lattice pattern.

In form growth starting from our first four imagined points in space, Figure C9 shows how all these patterns are interwoven.

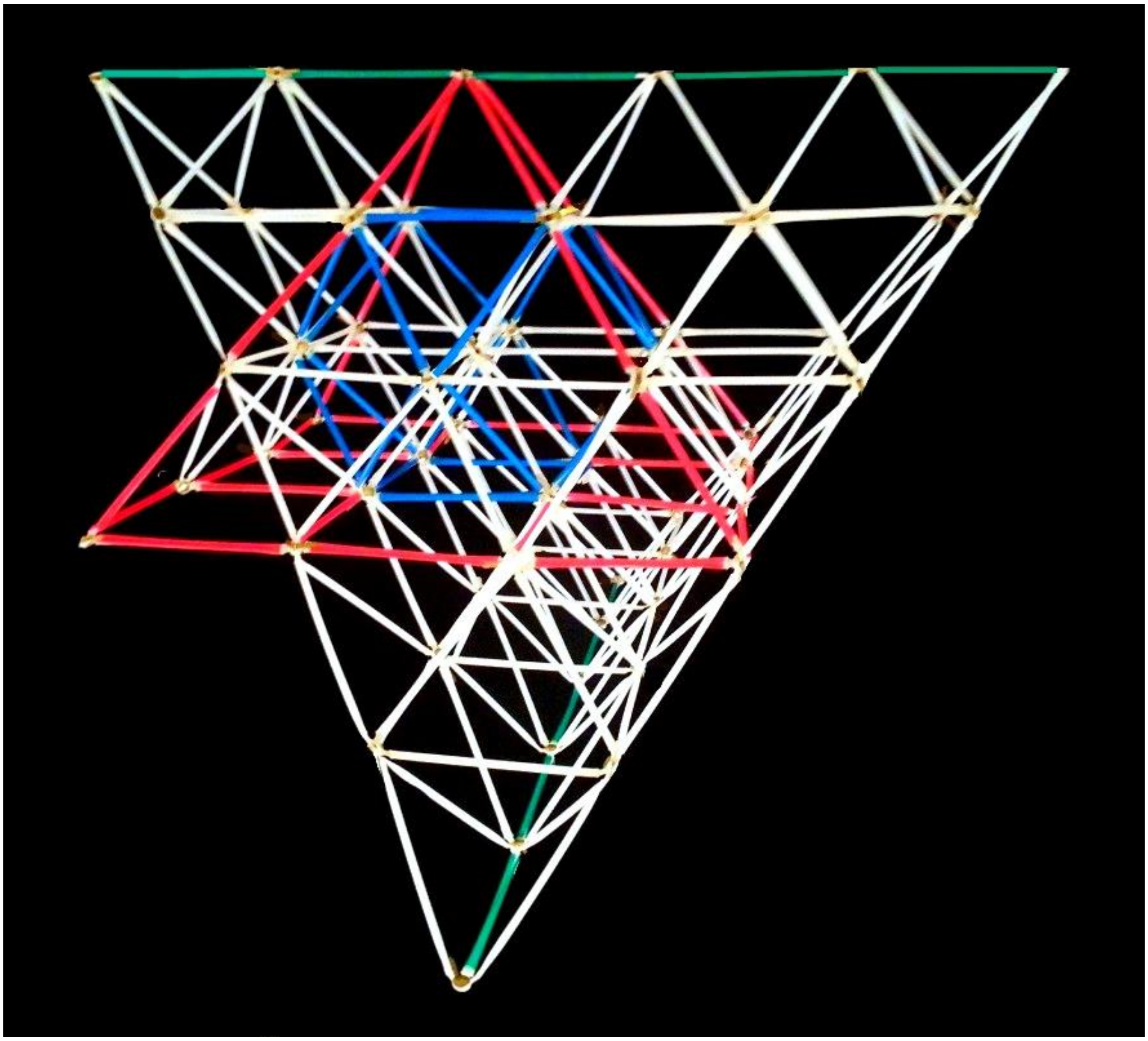


Figure C9: Tetrahedral-Octahedral Lattice with VE aspect highlighted

Tetrahedron growth is in White. The two lines in green are the orthogonal edges of the Tetrahedron. Highlighted within the lattice is a Pyramid form in red and a VE instance in blue. All of these forms can be seen (and in classical geometry are seen) as standalone forms, but it is more accurate I think, to see them simply as 'aspects' of Tetrahedral-Octahedral Lattice structure.

At this stage of buildout, we again witness alignment of the form to the three orthogonal (Cartesian) axes. In the Tetrahedron of Figure A10 they are formed by lines going through the mid-points of opposing edges. In an Octahedron, they are formed by lines pass through opposing nodes. In VE sub-form there are three pairs of opposing Square faces, highlighted in yellow in Figure C10. Lines passing through the mid-points of the opposing Square faces, shown in red in Figure C10, intersect at the center node of the VE and create three orthogonal axes.

Thus far in 3D growth, 'orthogonality' has morphed from line to node to face.

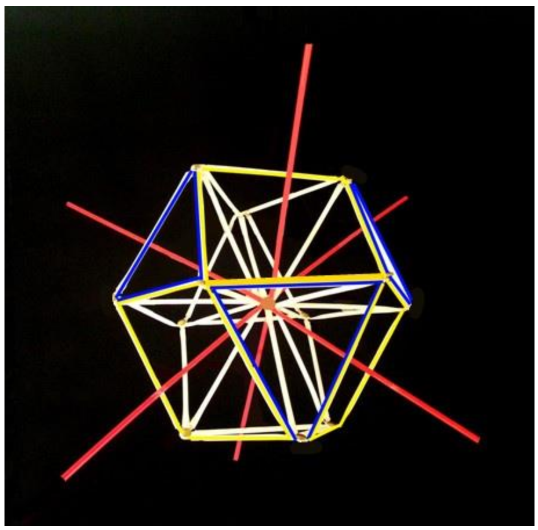


Figure C10: Orthogonal Square Faces of VE

The Icosahedron

There is much more to the Icosahedron than the twelve nodes and twenty triangular faces that its classical name implies.

As standalone forms both the VE and the Icosahedron have an outer layer of twelve nodes arrayed around a center point. However, they are fundamentally different forms. From a center node, the VE grows as a lattice to form an outer layer of twelve nodes, all at unit-length from the center node. In the Icosahedron, we have a constellation of twelve nodes that are unit-length distance from each other and balanced around a center but are not built out from a center node. We must consider the Icosahedron a distinct spatial intelligence – a '**Cage**'.

At first blush it appears that the center of the Icosahedron is unit-length distance to the twelve outer nodes and thus the Icosahedron can be derived from twenty regular Tetrahedrons, but this is not the case as the distance from the center to any point in the Icosahedron cage is slightly less than unit-length. This is explained in more detail in Appendix A: ‘Whole Number Dilemma’.

But all is not lost for whole number relationships. In progressive buildout from the Icosahedron the notion of unit-length Triangle and Square is still at play. Form growth that is balanced around a center is indeed whole number coherent - let’s explore.

Buildouts thus far have been linear (and by extension planar) as exhibited in Tetrahedral and Octahedral growth (*Cubic and Prism as well*). New nodes are simply added at unit-length along a linear path. There is no omnidirectional balance to Lattice growth. We see evidence of such Lattice growth in crystals and minerals.

With the Icosahedron, a new and distinct spatial intelligence is demonstrated, one that is perfectly balanced around a center, sphere-like. In the forming of an Icosahedron we discover the next great notion - that of ‘**Enfolding**’. The intelligence that folds a set of points perfectly balanced around a center is fundamentally different from the intelligence that creates Lattices. (*Intuitively, it seems to be a more sophisticated dynamic*.) Intelligence that folds around a center in a perfectly balanced manner also suggests a capacity for ‘spin’.

Figure D1 shows the formation of an Icosahedron via an enfolding process starting from five triangles in the upper left figure and continuing clockwise:

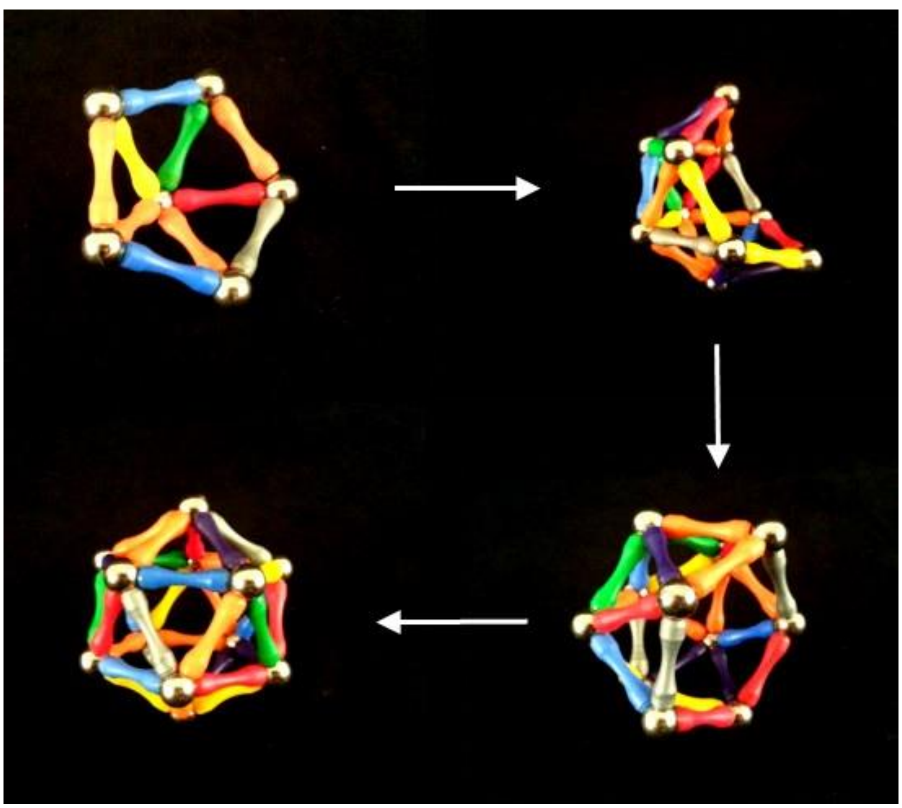


Figure D1: Points Enfolding into an Icosahedron

The completely formed Icosahedron, shown in Figure D2, is the starting point of continued exploration.

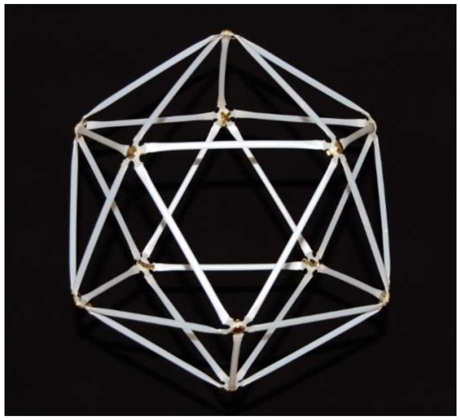


Figure D2: Icosahedron

Icosahedron Cages

Growth from the Icosahedron occurs in discrete layers - layers of concentric cages that are unit-length coherent at their faces. At each layer, the cage has more nodes and is spatially larger, yet each cage will always contain twelve, and only twelve, Pentagon faces. The increasing nodes count results from different patterns of unit-length Triangles or combinations of Triangle and Square faces filling in the area between the Pentagons.

*Per the analysis in 'Appendix A' we see that the distance **between** successive cages is slightly less than unit-length. To continue to model Icosahedral buildout, we use a '**Compensating**' modeling tool that allows us to form slightly 'squashed', (i.e., non-equilateral), Tetrahedral and Octahedral patterns between layers. This is a practical trick that helps us to find the defining pattern of the next cage layer.*

First Cage Layer

The first cage layer that comes into unit-length equilibrium around the Icosahedron is shown in red in Figure D3. The first cage layer is comprised of twenty unit-length Triangle faces and twelve unit-length Pentagon faces. The starting Icosahedron in the center is highlighted in blue.

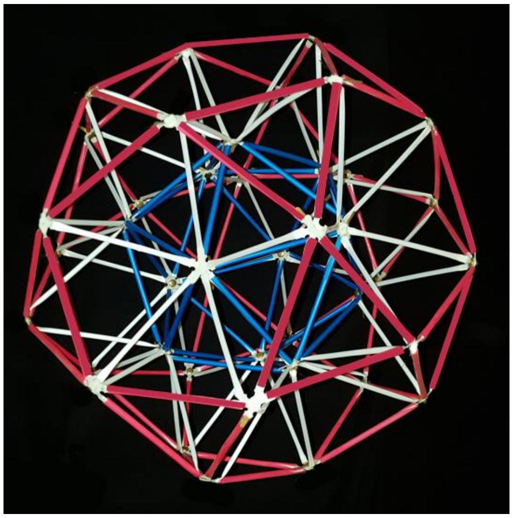


Figure D3: Model of First Cage Layer[5]

In Figure D4 the face pattern of the first cage is isolated by highlighting. A Pentagon face, one of twelve, is highlighted in yellow. It is surrounded by a pattern of five Triangles, highlighted in red. This is the face pattern of the Icosidodecahedron, an Archimedean solid.

It can also be appreciated that the line that runs along of each edge of the Pentagons form a 'great circle' whose plane runs through the center of the form.

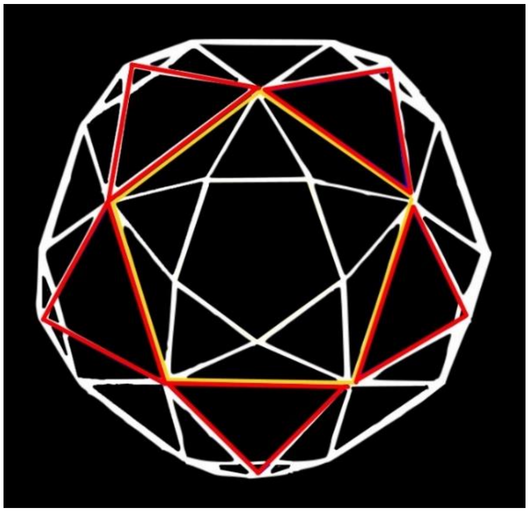


Figure D4: The Defining Pattern of First Layer Cage

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[5] *The white interstitial struts in Figure D3 are all equal length but slightly less than unit-length. As such, the entire model can be considered uniform but not regular. For purposes of modeling they serve as scaffolding that allows us to build outward to reveal the next cage pattern, which is regular*.

In the first cage layer around the Icosahedron at center, we observe the emergence of twelve distinct Pentagon faces. An important observation here is that Triangles alone, in a convex configuration, can create Pentagons. A direct progression from 3 to 5.

If we take another look at the starting Icosahedron at center, we can see the genesis of those twelve Pentagons -- they are subtly enmeshed in the Icosahedron. In Figure D5 a subset of three of the twelve Pentagons is highlighted - one in red, one in green and one in dashed-red. Just as we shifted our perspective to see the Octahedron, not as a set of Triangle faces but, as a set of orthogonal Squares, we can shift our perspective here, as well, and see the Icosahedron, not as a set of twenty Triangle faces but, as an enmeshed set of twelve Pentangles that in form growth (expansion) emerge as distinct Pentagon faces. The Pentagon is the 'signature' of the Icosahedron.

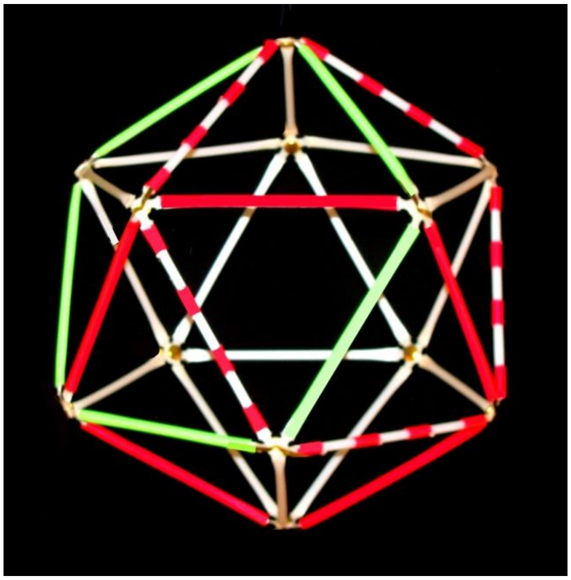


Figure D5: Enmeshed Pentagons of the Icosahedron

At this point we might pause and appreciate that the requirements to achieve omnidirectional coherency, as with Icosahedral growth, are more complex than for the linear/planar growth of Lattices. Growth outward from a center diverges in space whereas planar growth is spatially constant. Lattice growth is a regular pattern of unit-length Triangles and Squares. By contrast, in Icosahedron growth, unit-length coherency is only found at the faces of a given cage layer.

Now that we have seen the first Icosahedron cage and the emergence of the Pentagon faces, we might wonder if there is a more direct way to derive the Pentagon. After all, we have seen how the Triangle (3) of the Tetrahedron, when linearly oriented, creates the Square (4) of the Octahedron. Is it possible then that there is simple progression of '3 creating 4 creating 5' as a Lattice? In Figure D6 we perform a quick visual test of this premise.

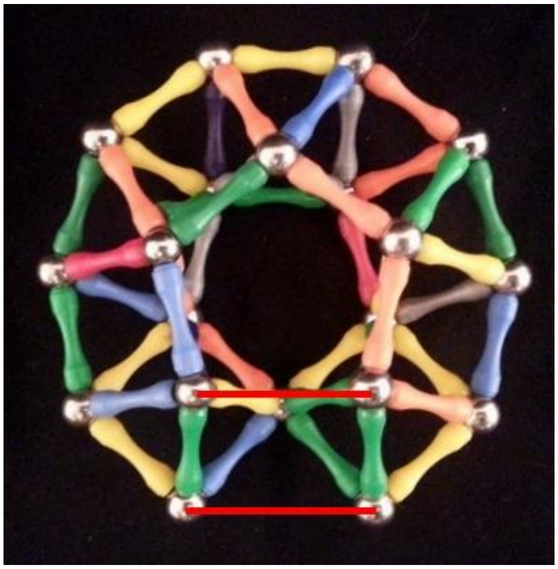


Figure D6: Octahedral Pyramids Do Not Create the Pentagon

Here a series of half-Octahedrons (i.e., Pyramids), composed of Triangle and Square, comes very close to encircling itself to create two Pentagons. But as can be seen by the highlighted red lines in the bottom Pyramid the gap is greater than unit-length, the pattern does not close, thus a Lattice progression is not possible.

Second Cage Layer

Continued buildout brings us the model shown on the left in Figure E1. The cage pattern is shown isolated on the right, it is the face pattern of the Archimedean Rhombicosidodecahedron solid.

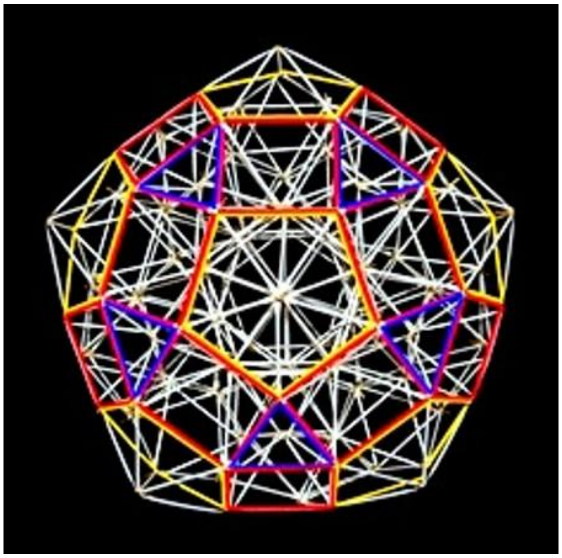
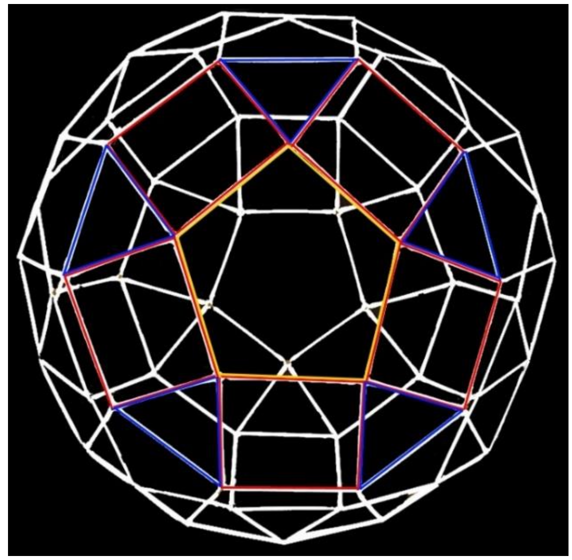


Figure E1 – Icosahedral model buildout highlighting the 2nd Cage layer.



The 2nd Icosahedral Cage layer isolated.

The 2nd cage's 'facet pattern' is isolated by color highlight in both images. There are five Squares shown in red, five Triangles shown in blue, creating the Pentagon shown in yellow. This is the face pattern of the Archimedean solid called the Rhombicosidodecahedron. But viewing this pattern through the lens of classical classification hides a rich significance, for here we may see this form as the dynamic of Triangle and Square creating the Pentagon. This is another important glimpse into the spatial intelligence of Nature. Recall in Figure B1 we observed, that when three nodes of two Tetrahedrons were arranged linearly, Triangles created the Square. Here, Triangle and Square, arranged in a convex pattern, create the Pentagon. An elegant progression of Triangle→Square→Pentagon (i.e., 3→4→5).

Figure E2 is a clear visualization of the Icosahedral expansion sequence thus far.

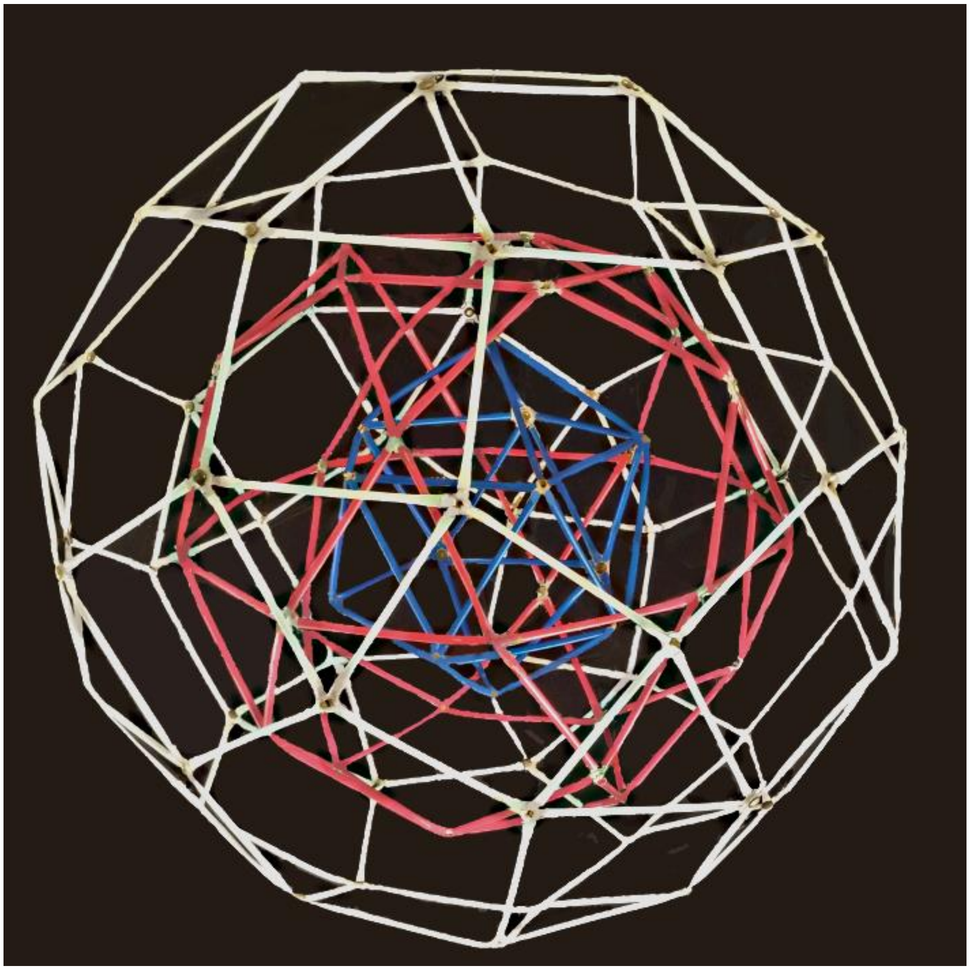


Figure E2: Icosahedral Expansion Sequence

At center in Blue is the initial Icosahedron. Expanding out we encounter, in Red, the 1st cage layer that has the face pattern of the Archimedean Icosidodecahedron solid. Expanding out again, we encounter, in White, the 2nd cage layer, that has the face pattern of the Archimedean Rhombicosidodecahedron solid. Twelve Pentagon facets in each layer are aligned along six axes emanating out from the center of the initial Icosahedron. Notice that in the first expansion from the Icosahedron to the first cage (from blue to red), the Pentagons are mated at each of their vertices. In the second expansion (from red to white), the Pentagons are mated at each of their edges.

The expansion continues with further buildout.

Third Cage Layer

With continued buildout the pattern of the third Icosahedral cage is revealed in Figure E4, highlighted in red. It is the Dodecahedron with edges of two unit-lengths. We can appreciate here that the Dodecahedron is a derivative of the Icosahedron, i.e., not a standalone Platonic form but an instance of Icosahedral growth.

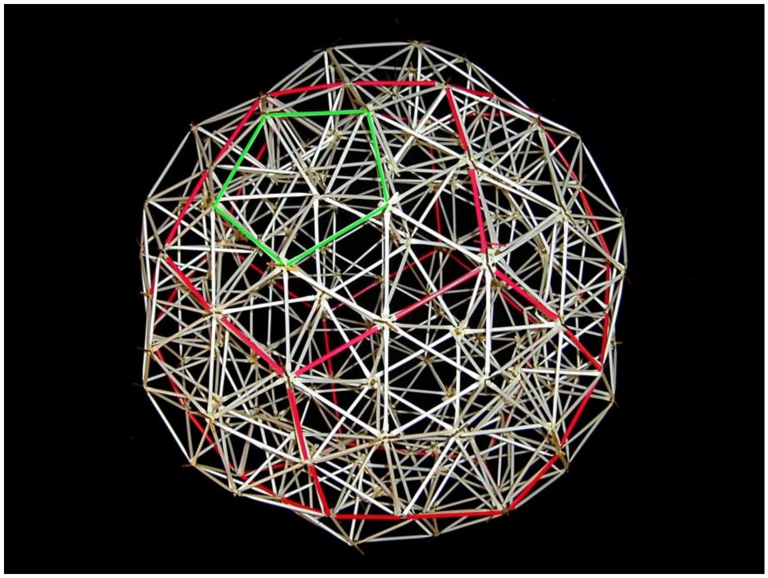


Figure E4: Cage Layer Three – Emergence of the Dodecahedron

Note the unit-length Pentagon, highlighted in green, projecting out from the center. This is one of twelve ever-present Pentagon faces of Icosahedral growth. Each side of the unit-length Pentagon has three Triangles that create the two unit-length edges of the Dodecahedron.

A similar pattern of three Triangles surrounding a unit Pentagon is found in the *Snub Dodecahedron*, an Archimedean Solid. However, the Snub Dodecahedron has only one row of Triangle separating the Pentagons, whereas here in Figure E4, there are two rows of Triangles, forming a slight folded Hexagon, separating the Pentagons. The 'fold' of the Hexagon forms one edge of the two-unit Pentagon, that in turn, form the cage of the Dodecahedron.

In Figure E4 you can also see that the unit Pentagon face, in green, is not on the same facet plane as the two-unit-length Pentagon in red. As such it is not a 'convex solid' like all Archimedean solids are. The two-unit Dodecahedron of Figure E4 is found in the concave trenches that surround the twelve unit-length Pentagons. The overall form, viewed as a solid, is like the non-convex *Truncated Small Stellated Dodecahedron*[6].

[6] The two unit-length Dodecahedron does not occur at the outer layer of the buildout like all the other cages do. Nevertheless, the underlying structure in the buildout creates linearly aligned edges of two unit-length, in the form of a Dodecahedron.

Fourth Cage Layer

The pattern of fourth cage is the 'Bucky Ball', highlighted in red in the model of Figures E5. You will no doubt recognize this pattern in the typical soccer ball. In Solid Geometry it is a *Truncated Icosahedron*.

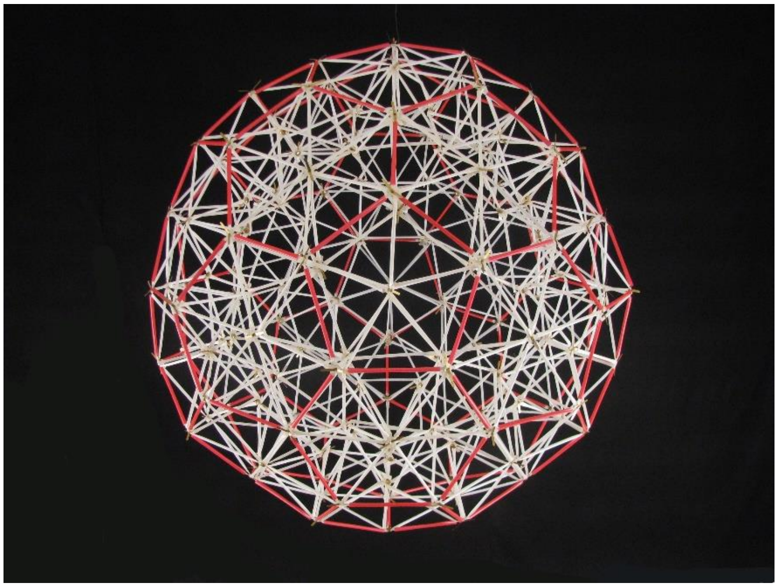


Figure E5: Fourth Cage Layer – The Bucky Ball

Like the form of Figure E4, each of the twelve unit-length Pentagon faces is surrounded by five Hexagon faces that, unlike Figure E4, are not folded and lay on the same facet plane as the Pentagon. In this sense the cages of Figures E4 and E5 are close cousins.

Characteristics of Icosahedral Cages

In the Icosahedral cage patterns that we have observed, the twelve Pentagons enmeshed in the starting Icosahedron at center expand and project outward along six axes. This was emphasized by the green Pentagon in Figure E4. This outward projection of twelve Pentagon faces at each layer of growth is a constant. That which changes at each cage layer is the pattern of Triangles or Triangle-and-Square combinations that fill the area between the Pentagons. Figure E6 shows the cage patterns of the first four cages of buildout from an Icosahedron at center. Each cage layer will have twelve such convex patterns inter-meshed. Each layer becomes more sphere-like as it grows and though the volume of the cage at each successive layer is larger, the struts of the cage faces are always unit-length; there are just more of them at each progressive layer.[7]

[7] We can see the general pattern of Icosahedral growth, consisting of twelve Pentagon faces within an ever-enlarging sea of Hexagons (Triangles), in Geodesic domes. A sea of Hexagons by themselves would form a flat plane. The Pentagons work to enfold to form. Geodesic Dome designs vary, most are based on the Icosahedron, though the facets are not necessarily equilateral.

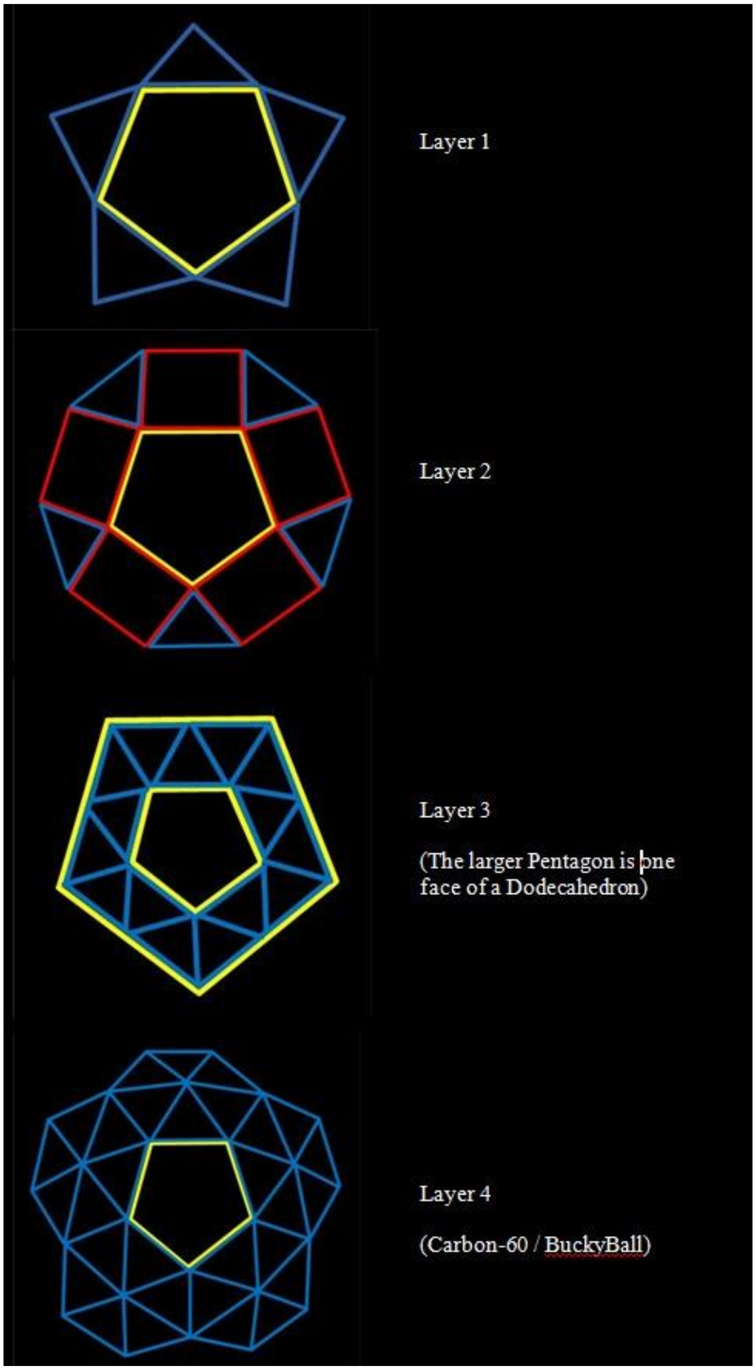


Figure E6: Defining Face Patterns of the Icosahedral Cages

(Although the Pentagons in the diagrams of Figure E6 appear to be of different size they are all the same size – unit-length.)

In Icosahedral buildout we observe successively larger Pentagon patterns emerging at each cage layer. Examples of repeating Pentagon patterns, the signature of the Icosahedron, are highlighted in red in Figure E7.

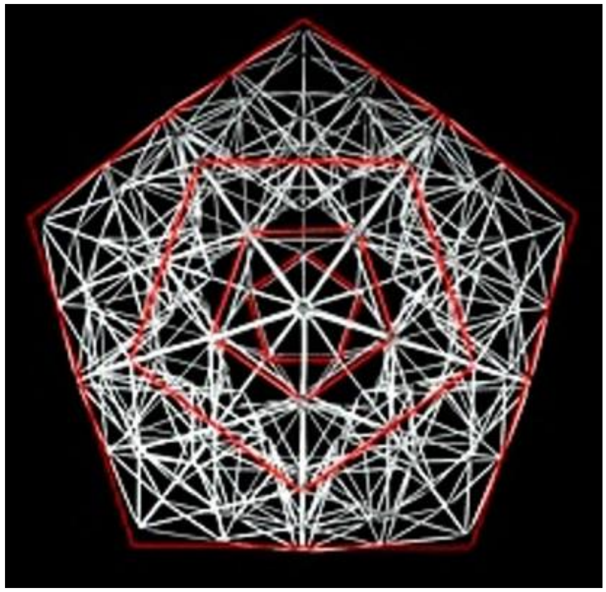


Figure E7: Successive Pentagons Emerging at each Cage of Icosahedron Buildout

At center, we can see the Pentagon of the inner Icosahedron with edges of unit-length, then a Pentagon of two unit-length and, at the outermost, a Pentagon of four unit-length. We also see this phenomenon in the Pentagons in Figure E4. Likewise, in the Lattice buildouts of Figures C3 and C4 we observe successively larger Tetrahedrons and Octahedrons appearing at each layer of planar growth. In these observations, we witness another fundamental and important notion -- '**Self-Similarity**', the notion of a pattern repeating itself at different levels of manifest growth. Synonyms for this notion, as it relates to the physicalizing process of nature, are '**Fractal**', '**A correspondence of resemblance from inner to outer'**, as well as the old favorite, **'As Above, So Below'**.

Self-Similarity does not mean strict pattern exactness at every level of growth. But, as everything at the surface of the physical world is a geometric form ("*God geometrizes continually*" - *Plato*), everything can be understood as a chain of geometric patterns, guided by fundamental patterning notions found in any given portion of that chain. A simplistic example of Self-Similarity is where in the architecture of a brick building, we can still recognize the form of the brick that it is built with.

(Additional thoughts on Self-Similarity are presented in Appendix B.

Convex Facets

Inspired by the beautiful convex facet pattern of the second Icosahedral Cage of Figure E2 (i.e., the Rhombicosidodecahedron), this section explores all convex node patterns (i.e., dome-like facets) comprised of Triangle and Square at the outer edge. The set of all possible regular facets are those with a Triangle, Square, Pentagon and Hexagon at center as shown in Figure E9.

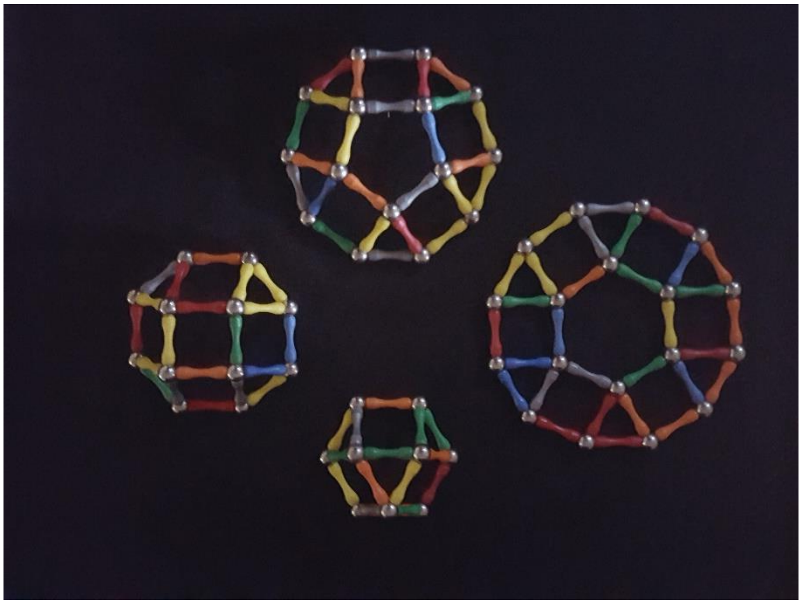


Figure E9: Convex Face Patterns of 3-4-5-6

What all these facet patterns have in common is that the unit-length Triangles and Squares that surround the center shape create a set of outside nodes that all lie on a plane, i.e., the outer edge lies flat.

The facet at the bottom with the Triangle at center is a six-sided plane. Moving clockwise around the Square is an eight-sided plane, around the Pentagon is a ten-sided plane, and around the Hexagon is a twelve-sided plane. The Triangle-centered pattern, at the bottom, is the most convex form. As we progress to Square and Pentagon the overall pattern becomes shallower ending with the Hexagon-centered pattern that is perfectly flat. We thus observe that only three of these facets, when fully inter-connected, creates a Cage. The Hexagon-centered pattern cannot. Beyond the Hexagon-centered pattern, no facet pattern comprised of Triangle and Square is possible.

When we mate convex facets of Triangle, Square and Pentagon we get Cages. Mating two of the Triangle-at-Center facets we get the outer form of the Cuboctahedron shown in Figure E10.

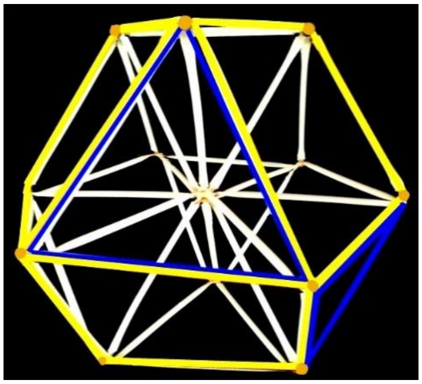


Figure E10: Mated Facets of Triangle-at-Center (Cuboctahedron)

Mating twelve Pentagon-at-Center convex patterns we get the second Icosahedral cage, the Rhombicosidodecahedron, seen earlier in Figure E2 and repeated below in Figure E11. It is composed of twelve of these patterns inter-meshed.

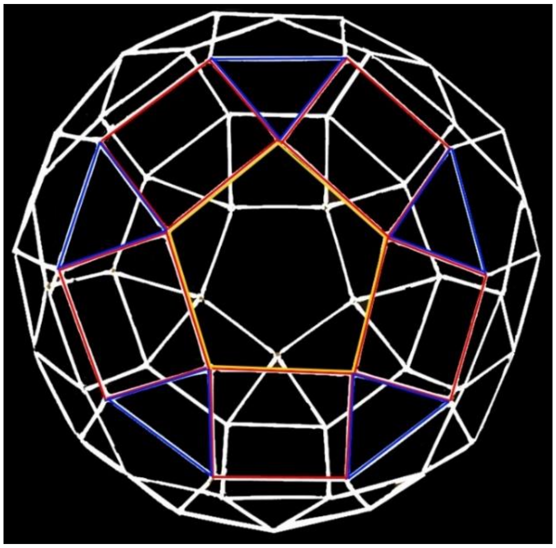


Figure E11: Mated Pentagon-at-Center Facets

Lastly, in Figure E12 we see three orientations of the Square-at-Center convex facets creating a cage. The middle form is the Archimedean 'Rhombicuboctahedron'.

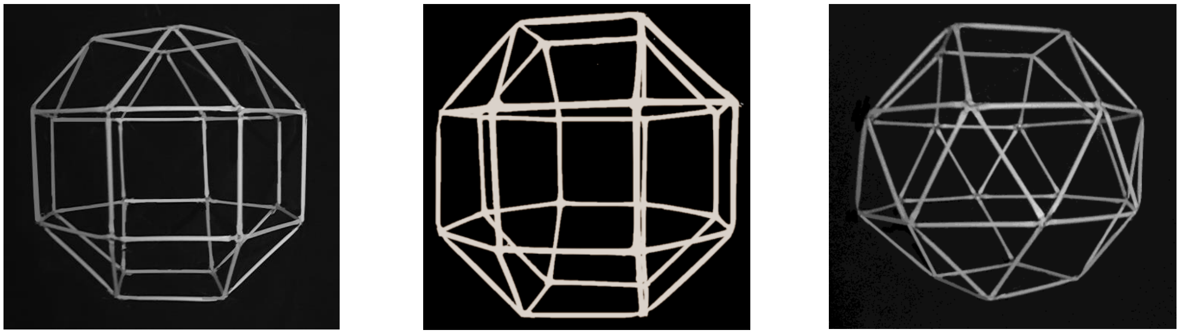


Figure E12: Cage Variations of the Square Convex Pattern

Figures E10 and E12 show us that it is not only the Icosahedron (Pentagon) that produces a cage. In fact, the node pattern of any given polyhedron form can be considered a cage. However, some cages are dead-ends that will not allow further buildout to create a successively larger cages with unit-length coherency (i.e., Triangle and Square), as the Icosahedral cages do. For example, the

Stellated Dodecahedron in Figure E13 will not allow further unit-length buildout. In this sense, all stellated forms are dead ends.

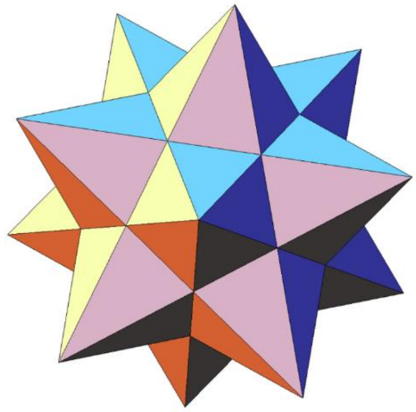


Figure E13: Stellated Dodecahedron

Tubes

The 'Enfolding' principle that produces the omnidirectional Icosahedron can also fold along a linear axis to produce an Icosahedral '**Tube**' shown in Figure F1[8]. This tube consists of unit-length Triangles that travel along a linear axis and intersect to create a series of Pentangles that are perpendicular to the line of travel. Seen from the end, the profile of this tube is the Pentagon.

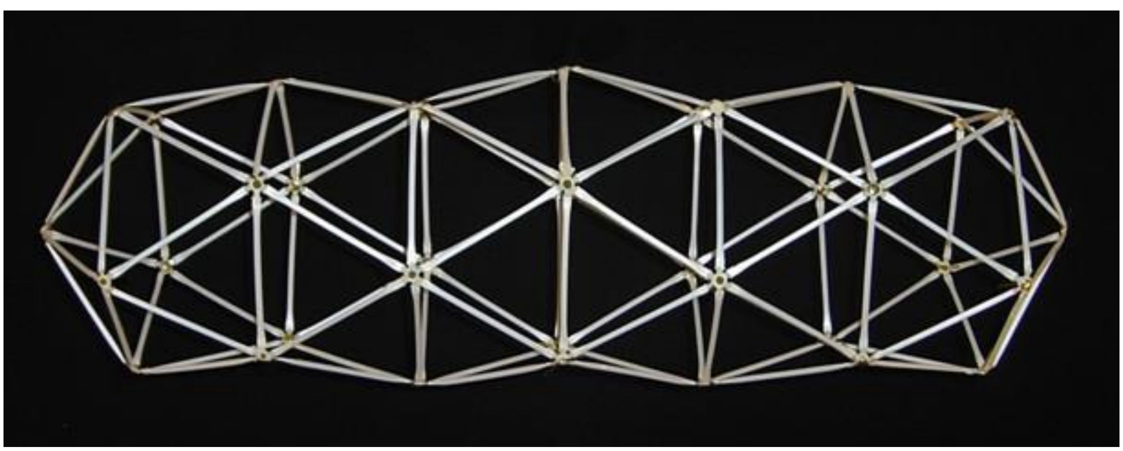


Figure F1: Icosahedral Tube

Tubes suggest again the notion of 'Linearity', i.e., a travel of points along a straight line. By contrast, Icosahedron cages that enfold around a center suggest the notion of spinning. In growth,

[8] *Tubes may also be referred to as Prisms or Anti-Prisms*.

cages become more sphere-like at each level, tubes simply get infinitely longer as they travel along an axis.

It should be appreciated that Icosahedral tubes can intersect with Icosahedral cages to form complex combinations of polyhedral nodes with tubes as branches. Figure F2 models an Icosahedral tube intersecting with an Icosahedral cage. Here, it is as if the tube travels through the cage.

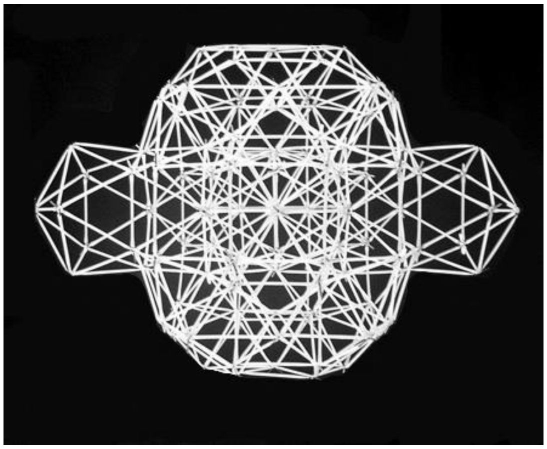


Figure F2: Linear and Spherical Growth of Icosahedron

We observe that opposing nodes of the Icosahedron create lines through the virtual center that represent axes of potential linear growth. The axes emanate outward omni-directionally from the center and diverge in space. The red line in Figure F3 shows just one of six axes going through the virtual center in yellow.

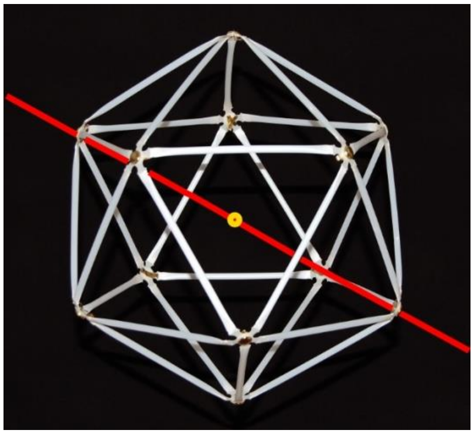


Figure F3: Axis of the Icosahedron

Figure F4 shows all six axes (or twelve rays) in red emanating out from an Icosahedron at center. Icosahedral Linear Tubes can travel along these axes. *(One axis cannot be seen because it is coming straight out at you.)*

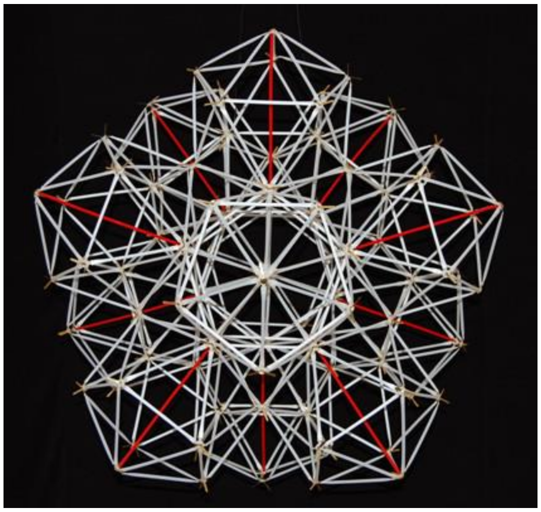


Figure F4: Six Axes of the Icosahedron

With the Linear Tube Notion in mind we can now appreciate other tube profiles. As seen earlier in Figure A6, the Tetrahedron cannot fill space omni-directionally off its faces, however, it can grow linearly along its faces as a ‘twisting tube’ shown in Figure F5. The twist is highlighted by the red, green and blue struts. At first blush it appears that the form spirals as it grows but its growth is straight and perfectly balanced around a centerline inside the tube. It is comprised solely of Triangles. Its profile is the Triangle.

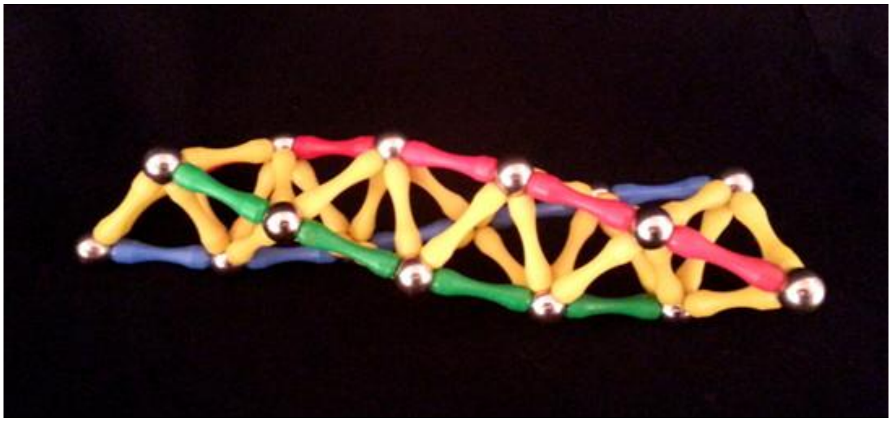


Figure F5: Tetrahedral Tube (Helix)

The tube of Figure F6 has a profile of the Square. Squares, shown in gray, rotate 90º at each successive layer of travel.

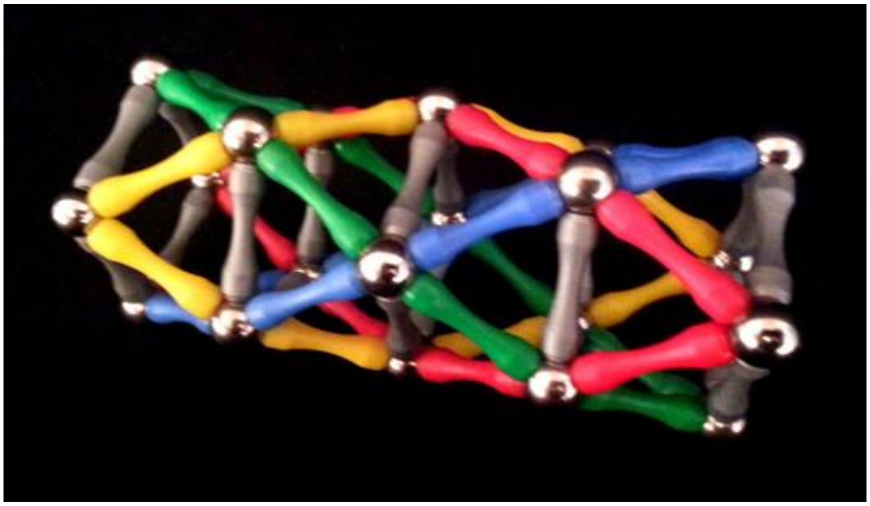


Figure F6: Tube of Square Profile

We've already seen the Pentagon tube in Figure F1. Figure F7 shows a series of tubes with profiles of six, seven and eight.

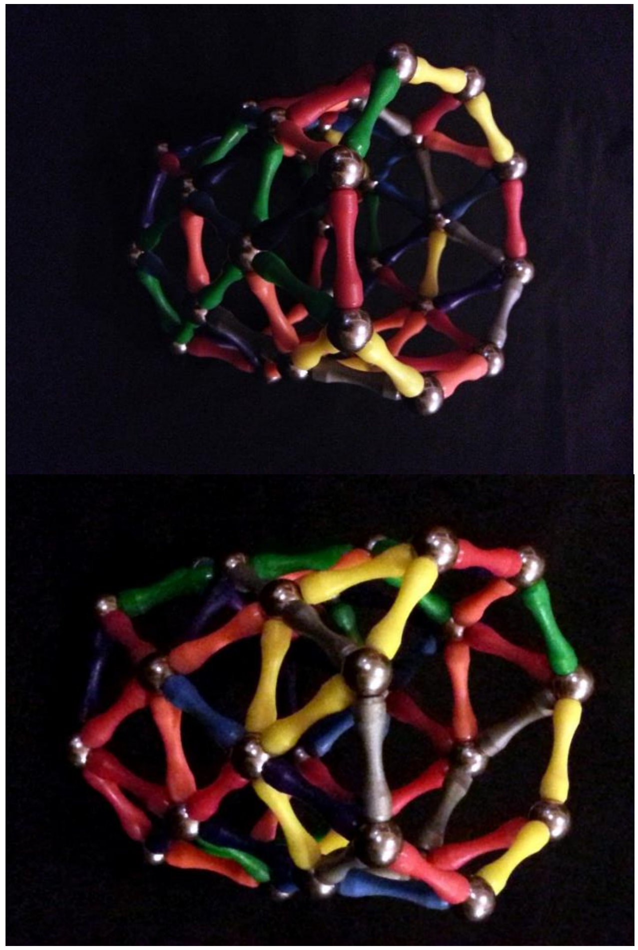


Figure F7: Tubes with Profiles of Six, Seven and Eight Respectively

Any profile number is possible. With each increase in the value of the profile the tube wall becomes less convex. As the profile approaches infinity, the wall pattern approaches the flat 2D ‘Flower of Life’ pattern. Amazingly, the inter-meshed patterns of Triangles can articulate to any degree of curvature.

Though not modeled, it should be noted that Square prisms can be constructed but would not be stable. Whereas Triangle-based tubes have a ‘spiraling’ travel of inter-meshed nodes, nodes of a Square-based tube would form straight lines parallel to the direction of tube travel and orthogonal to the plane of the tube profile. Whereas the Triangle tube possesses strength in terms of both tension and compression between all nodes, the Square tube does not. This weakness probably makes this tube form impossible in Nature.

Triangle-based tubes can be thought of as a parallel set of nodes in the form of a Spiral.

As we end our discussion of Tubes, consider that the Tube forms above have Triangle faces and varying end profiles, Lattice forms have Triangle and Square faces, and Cage forms can have

Triangle, Square and Pentagon faces. As such, off any given face of a Lattice or a Cage there is the opportunity for a Tube of the same profile to integrate into it and interconnect between multiple forms. Such interconnected forms are called labyrinths. Figure F2 is an example of a tube intersecting with an Icosahedron-based form.

Plane Wonders of Triangle and Square

If we were to unfold all the tubes described above, they would flatten out into a 2D plane of unit triangles shown in Figure P1.

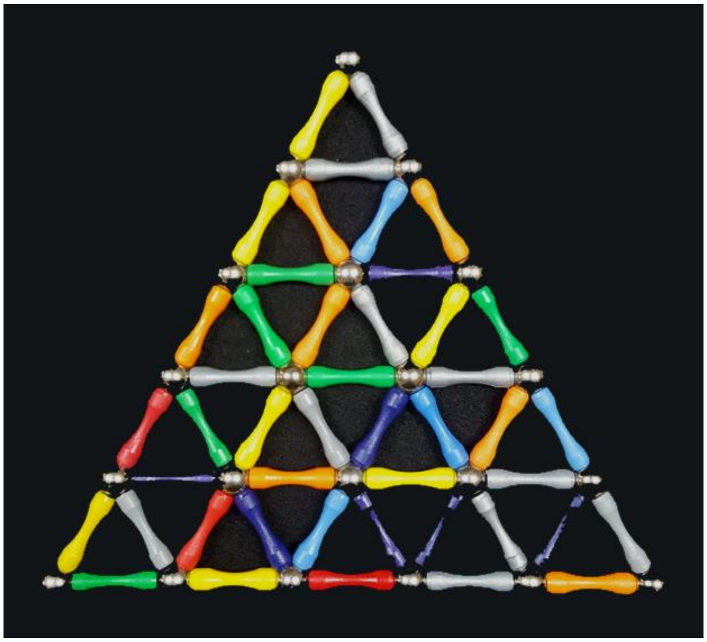


Figure P1: Plane of Unit Triangles

And important aspect of this 2D unit Triangle plane is the linearity of the nodes. As can be seen, the node pattern comprises three sets of lines with a separation of 60º that can grow infinitely long. In Figure P2 a 2D plane of Squares is shown.

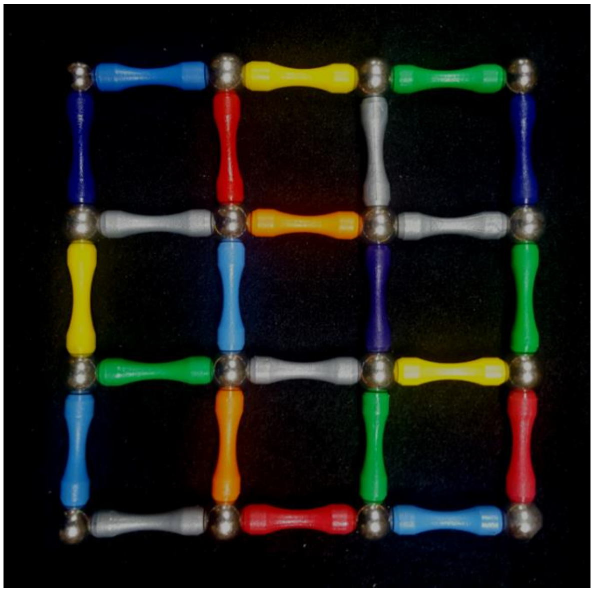


Figure P2: Plane of Unit Squares

Here we see two sets of lines at a separation of 90º. The common characteristic of infinite linearity allows these two planes types to mesh together and form 3D Lattices such as the Tetrahedral-Octahedral Lattice of Figure C9 (re-shown below).

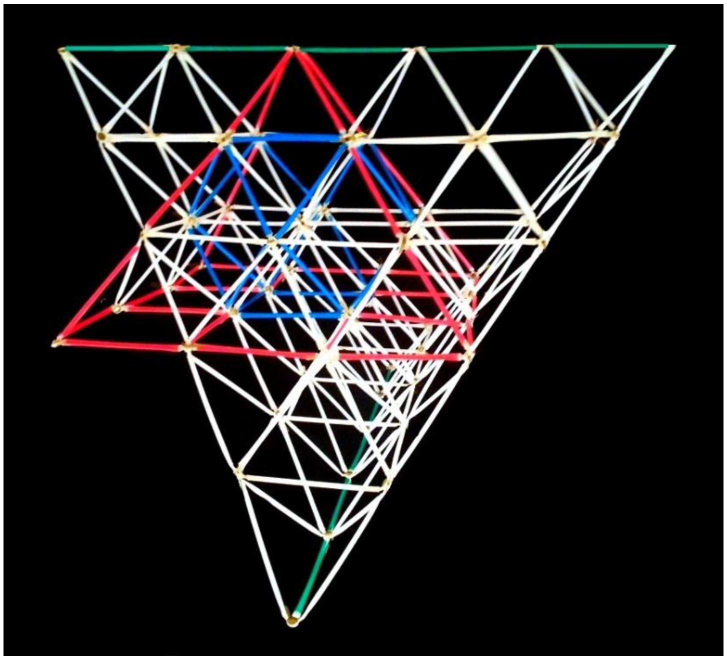


Figure C9 (copy) – The 3D Tetrahedral-Octahedral Lattice

Just as the lines of both homogeneous 2D planes can travel infinitely, the 3D Lattice of Figure C9 can grow infinitely large.

Less obvious is a third plane pattern, a heterogeneous plane that contains both unit Triangle and Square as shown in Figure P3. As it is so common to see just homogeneous Triangle or Square planes, it is almost magical when you realize that unit Triangle and Square can also mesh to form a 2D plane.

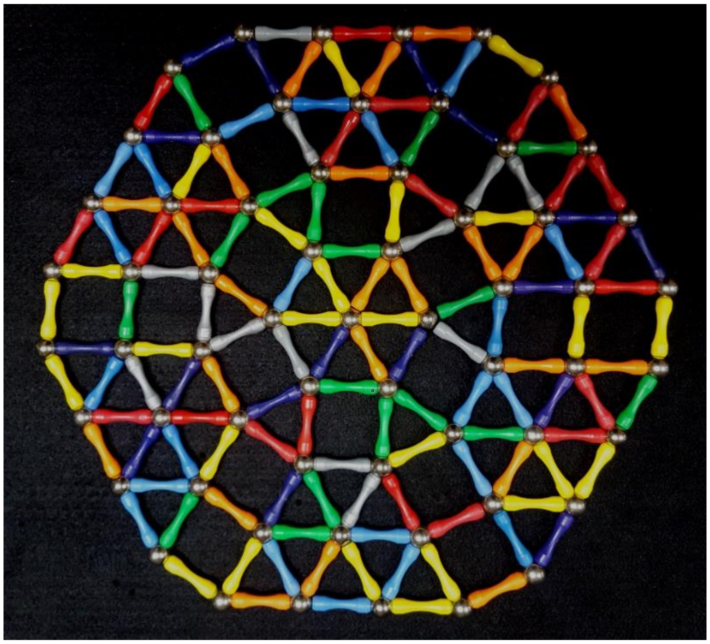


Figure P3: Plane of Triangle AND Square

A basic axiom of analytical Geometry is that the angles around each node must add up to 360º.

In Figure P1 the vertex angle of the 6-unit Triangles at each node is 60º, thus 6x60º = 360º.

In Figure P2 the vertex angle of the 4 Squares around each node is 90º, thus 4x90º = 360º.

In the heterogeneous plane of Figure P3 the angles around each node add up to 360º as well, either as (6x60º) or as (3x60º + 2x90º). In the latter case there are two sub-patterns. Figure P4 shows the sub-pattern where the two Squares are separated by a Triangle. This sub-pattern promotes circular growth, whereas the sub-pattern of Figure P5 promotes linear growth. Both sub-patterns are found in Figure P3. In all cases, the angles around every node of Figure P3 add up to 360º.

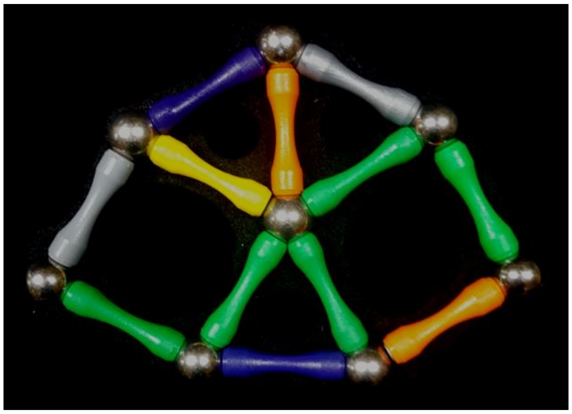


Figure P4: Circular Sub-pattern of Hybrid Plane

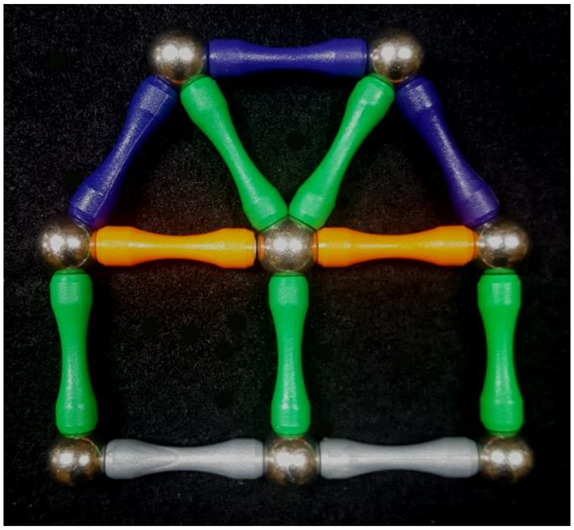


Figure P5: Linear Sub-pattern of Hybrid Plane

The top portion of the plane of Figure P6 shows the linear growth pattern off the heterogeneous plane of Figure P3.

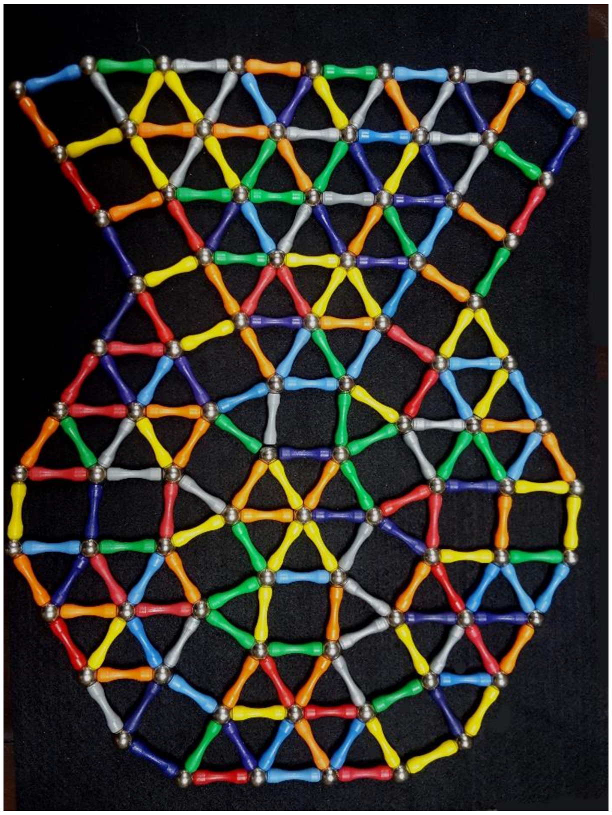


Figure P6: Linear Growth Pattern of Hybrid Plane

By contrast a circular growth pattern is an infinite array of encircled hexagons, a set of three is shown here in Figure P7.

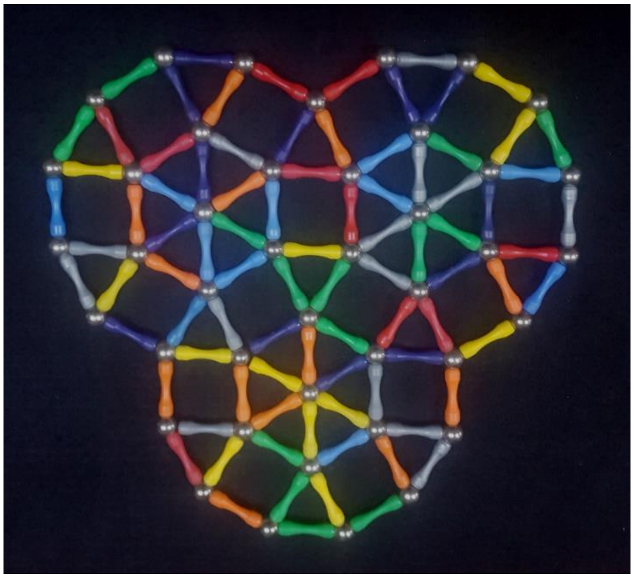


Figure P7: Circular Growth Pattern of Hybrid Plane

In Figure P7 we observe recurring hexagon patterns. Recalling our earlier discussion about the Vector Equilibrium (VE), we know that the VE is comprised of four hexagons, per Figure C6. As such, it is enticing to wonder if the hexagons of the heterogeneous plane may be VE forms in 3D that somehow create a different Lattice pattern than the 'linear' Lattice pattern of Figure C9. An exploratory buildout in Figure P8 dispels this possibility.

Figure P8: Failure to Enclose a VE in the Hybrid Plane

In Figure P8 a VE is built out of one of the hexagons which is then encircled by a combination of Triangles and Squares (lying flat). An attempt to repeat the encircling of another hexagon in the VE is a failure, for as can be seen in Figure P8, at the point where the second encirclement meets the first encirclement, the nodes do NOT line up, i.e., a cleave plane does not exist.   In a successful mesh the two encircling planes would share nodes. In this case the second plane can be seen to pass through the interstice between the nodes of the first plane.

From this observation we can more fully appreciate that the possibility of infinite 3D Lattice growth is enabled by the infinite linearity that characterizes the node pattern of 'homogeneous' Triangle and Square 2D planes. Both homogeneous plane types have infinite extent of node linearity in common, they simply differ in the angles of their linearity. The node pattern of the heterogeneous plane does not possess infinite linearity and thus cannot mesh into a 3D Lattice. For this same reason the heterogeneous plane also appears to not fold into a tube.

The Lessons of Geometry

Let's summarize our journey thus far:

We began with a thought experiment that worked with imagined points in space. Once beyond a few points it became difficult to mentally visualize point arrangements, so we augmented our thought experiment with node-and-strut model building. Observations made in the progressive buildout of models led us to a more sophisticated understanding of Nature's patterning tricks.

A key premise of our thought experiment was that all points are equal in terms of force of attraction and repulsion, resulting in equal space between them, a line of unit-length distance. As we added points the patterns grew volumetrically in three dimensions to create complex patterns of nodes that were relative to unit-length and regular. The first four points created the first regular 3D form, the Tetrahedron, where we observed that each pair of points forms an orthogonal 'crisscross' of unit-length lines with the other pair. In further buildout, this crisscross, through a force-balancing dynamic that linearized the points, surfaced the Square of the Octahedron. Here lay the nascency of the notion of **'Duality'** of Triangle and Square, i.e., the notion that it takes Triangles and Squares to coherently fill 3D space, either as a Lattice or a Cage.

In the buildout of Lattices, we observed Triangle and Square becoming intersecting **planes** of Triangles and Squares. In the Icosahedron we observed the '**Enfolding**' principle that creates Icosahedral Cages - forms that expand out from a center point as a succession of unit-length cages comprised of Triangles, Squares and Pentagons, a center-balanced enfolding. We also observed nodes enfolding around a linear axis forming Tubes of infinite profile size and infinite length.

We find that Icosahedral growth is fundamentally different than Lattice growth. As a general observation, there are only two kinds of growth patterns – **Linear** (Planes, Lattices) and **Enfolded** (Cages, Tubes).

In the broadest possible abstraction, an enfolded pattern is that which turns inward becoming distinct and particle-like with a center. A lattice is that which extends outward infinitely into dimensional space with no unique center.

Throughout the progression of building from simple to complex we observed similar patterns of Triangle, Square and/or Pentagons repeating themselves at successive layers of growth. This is the notion of **Self-Similarity** in Nature's form making.

The most potent observation, however, has to do with our initial premise: The implicit assumption that all points are **equal in terms of their force of attraction and repulsion** in 3D space. Without this assumption, we would be faced with an infinity of possible point strengths, resulting in an infinity of possible point separation distances, and thus an infinity of irregular form[9]. It would be extremely difficult to comprehend 3D spatial patterning in this context. However, there is just one, and only one, context where all point strengths are equal[10], and in this context, we see that **equality of point force** implies the spatial equilibriums of **extent** (i.e., unit-length distance of separation) and **direction** (i.e., orthogonality)[11]. Thus, in the premise of point equality we find the most abstract definition of Duality: The unit-length Triangle is the spatial representation of the **'idea of equalized extent'**, the unit-length Square is the spatial representation of the **'idea of equalized direction'**[12].

Considering that the Square can be derived from the linear orientation of Tetrahedrons per Figure B1 (i.e., Triangles creating the Square), and that the Pentagon and Hexagon patterns can be derived from Triangle or Triangle-and-Square arrangements, we can appreciate then, that from the base premise of **point equality** flows **all** regular pattern.

Seeing the Forest for the Trees

Let's return to model building but now with a focus on common assumptions and perceptions about physical form, and how model building can cast a new light on incumbent perceptions and assumptions and possibly shift our perspective. (*per Prologue*)

In 3D modelling it is extremely important to be able to highlight specific node patterns within a large network of nodes because it is extremely easy to miss them. In the Tetrahedral-Octahedral Lattice it is easy to spot the planes of Triangles and Squares, the Tetrahedrons and the Octahedrons. Those patterns seem to steal our attention to such an extent that we cannot see more complex sub-forms, i.e., node constellations, within the Lattice model. We noticed this 'hiding in plain sight' phenomenon with the Cuboctahedron (VE) form in Figure C9. It is not until you highlight specific strut patterns in a different color that the sub-form 'pops out' from the mesh. Many sub-forms exist within the infinitely space-filling Tetrahedral-Octahedral Lattice. In modeling, as the Lattice grows, the number of nodes increases, and larger, more complex, sub-forms come into existence. That said, all sub-forms within the Lattice are

[9] *Fields of study that seek space-filling Geometries relax the requirement of equality to a small degree. This results in finding practical space enclosures in building architecture.*

[10] *We also find this notion in Arithmetic where there is the implicit assumption that the order of natural numbers is equally spaced. That is, the difference between each number is identical throughout the entire number space. There are no 'privileged' numbers.*

[11] *This is the essence of the term 'Vector Equilibrium'. A vector comprehends both magnitude of distance and direction. It should also be appreciated that these aspects correlate to the two ways that spheres can be packed, 'closest' and 'cubic' packing.*

[12] *See Appendix C.*

connected and thus can also infinitely fill space. We should therefore recognize a given instance of a sub-form as simply an **'aspect'** of the Tetrahedral-Octahedral Lattice.

A significant sub-form hidden within the Tetrahedral-Octahedral Lattice is the √2-Cube. Figures Q1 and Q2 show a subset of Cubes (in red) in the form of three orthogonal branches of a √2 Cubic Lattice. From this highlighting we can see that the √2 Cubic Lattice in red is really a '**co-Lattice**' of the unit-length Tetrahedral-Octahedral Lattice in white. A network of Tetrahedrons must come first to create the pattern of Octahedrons to then create the √2-Cubes.

The highlighted models in Figures Q1 and Q2 make evident two significant aspects of the single Tetrahedral-Octahedral Lattice, a relationship that is exceedingly difficult to see in an un-highlighted model. The red struts were not part of the original Tetrahedral-Octahedral Lattice model, they were added in afterwards to connect the diagonal nodes of the native Octahedrons. They show us how a latent node pattern, the √2 Cubic Lattice, so obvious once highlighted, can be 'hidden in plain sight' to our perception when not highlighted.

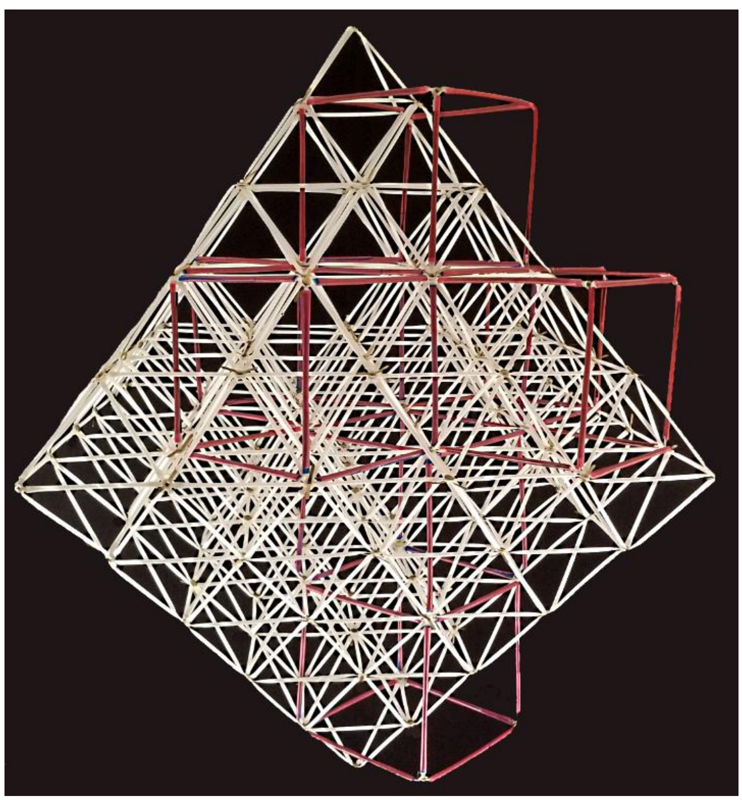


Figure Q1: √2 Cubic Lattice Aspect – Looking at Octahedron Edge

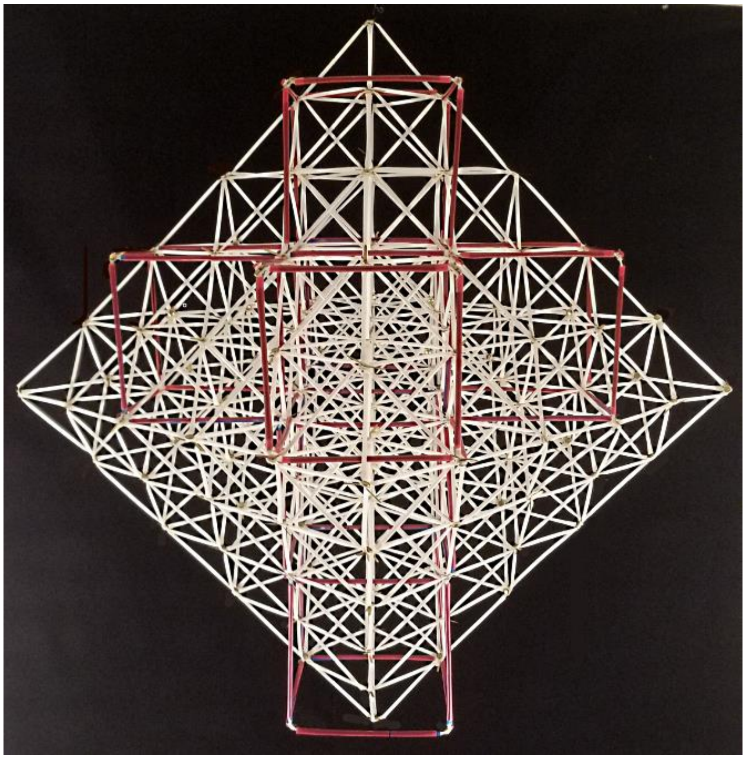


Figure Q2: √2 Cubic Lattice Aspect – Looking at Octahedron Vertex[13]

We alluded to the √2 Cube earlier in Figure B4 which shows a single instance of the √2 Cube. In Figures Q1 and Q2 we now see the full context and how the √2 Cubic Lattice is intrinsic to the Tetrahedral-Octahedral Lattice.

It is generally accepted that a unit-length Cube pattern is not a stable 3D form because spatial stability requires node Triangulation. In the Tetrahedral-Octahedral Lattice the √2 Cubic Nodes are indeed Triangulated, just as we see in Figure B4. From this observation we can probably conclude that, in Nature, stable cubic forms exist. They are 'aspects' of an underlying Tetrahedral-Octahedral Lattice … and the powerful notion of the Triangle-Square Duality.

Regular geometric form holds many surprises in terms of spatial elegance. Looking closer inside any given √2 Cube of the Tetrahedral-Octahedral Lattice, such as in Figure Q3, we see a characteristic pattern of the intersecting Triangle planes.

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[13] *Only one √2 Cubic Lattice is shown in Figures Q1 and Q2, but there are actually three distinct √2 Cubic Lattices within the Tetrahedral-Octahedral Lattice. A given √2 Cube has twelve √2 edges (i.e., struts, or lines of force between nodes). In the Tetrahedral-Octahedral Lattice these edge/struts are formed by the √2 diagonal of each of twelve neighboring unit Octahedrons. Given that there are three diagonals in each Octahedron, there are thus three distinct Cubic Lattices, spaced √2/2 apart. The struts of each of the three distinct Cubic Lattices intersect at their midpoint in the center of each Octahedron (this is not modeled in Figures Q1 and Q2).*

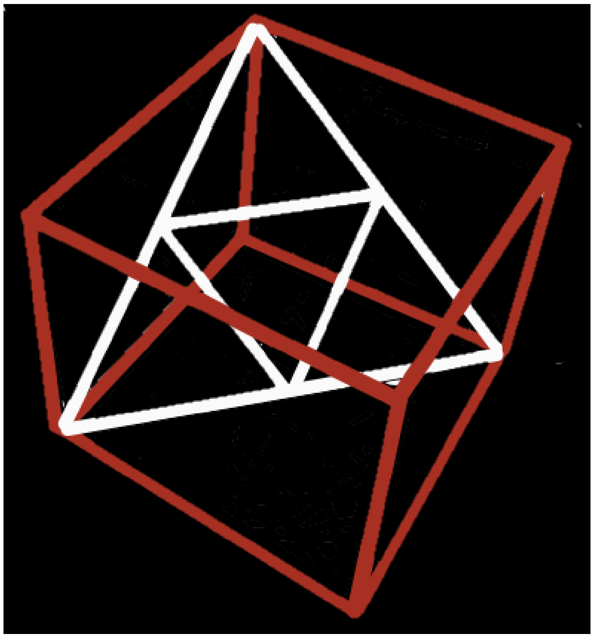


Figure Q3: Characteristic Triangle Plane within a √2 Cube

It is a Triangle of 2-Unit length composed of 4 unit-length Triangles - **a visual description of the relationship of Triangle to Cube.** Eight of these 2-Unit Triangles emanate from each cube vertex and intersect. The center Triangles compose the sides of the Octahedron at center as shown in green in Figure Q4.

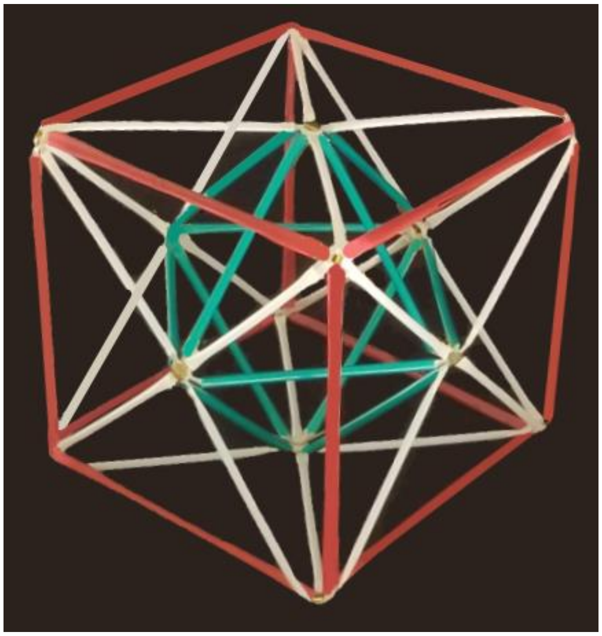


Figure Q4: Tetrahedrons and Octahedron in relation to the √2 Cube – Isometric View

Each face of the √2 Cube is formed by the planar crossing of two unit-lengths, evident in Figure Q5. The center points are the vertices of the Octahedron in the center of the Cube. This is the characteristic pattern of the √2 Cube within the context of the Lattice.[14]

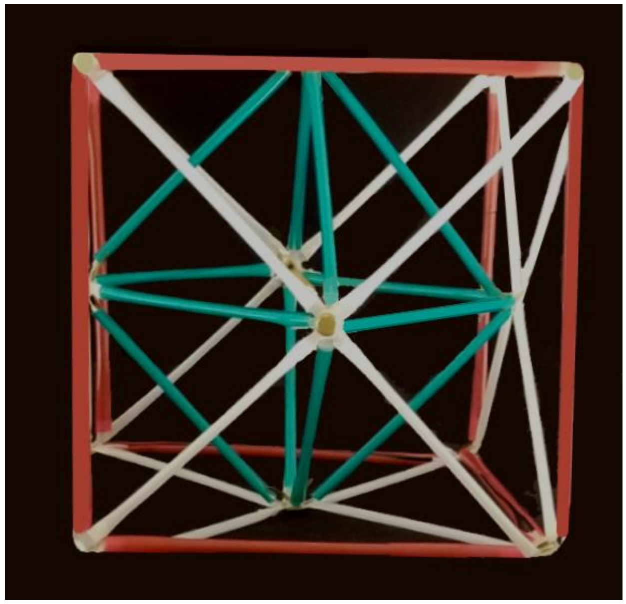


Figure Q5: Characteristic √2 Cube Created by Unit Crosses – Side View

One can easily imagine an infinitely large Tetrahedral-Octahedral Lattice constructed by simply stacking together an infinite number of this building block. A 'block stacking' visualization is an easy and highly useful way to perceive the Tetrahedral-Octahedral Lattice compared to imagining intersecting planes of Triangles and Squares as in Figures C3 through C5.[15]

Figure Q6 shows a stacking of four Characteristic √2 Cubes within a Tetrahedral-Octahedral Lattice.

[14] *Figures Q4 and Q5 provide a good visual description of the concept of 'duals' in Solid Geometry. The Cube and the Octahedron are duals.*

[15] *We seem to have an instinctual inclination, like proprioception, toward imagining 3D form in terms of Right-Angles.*

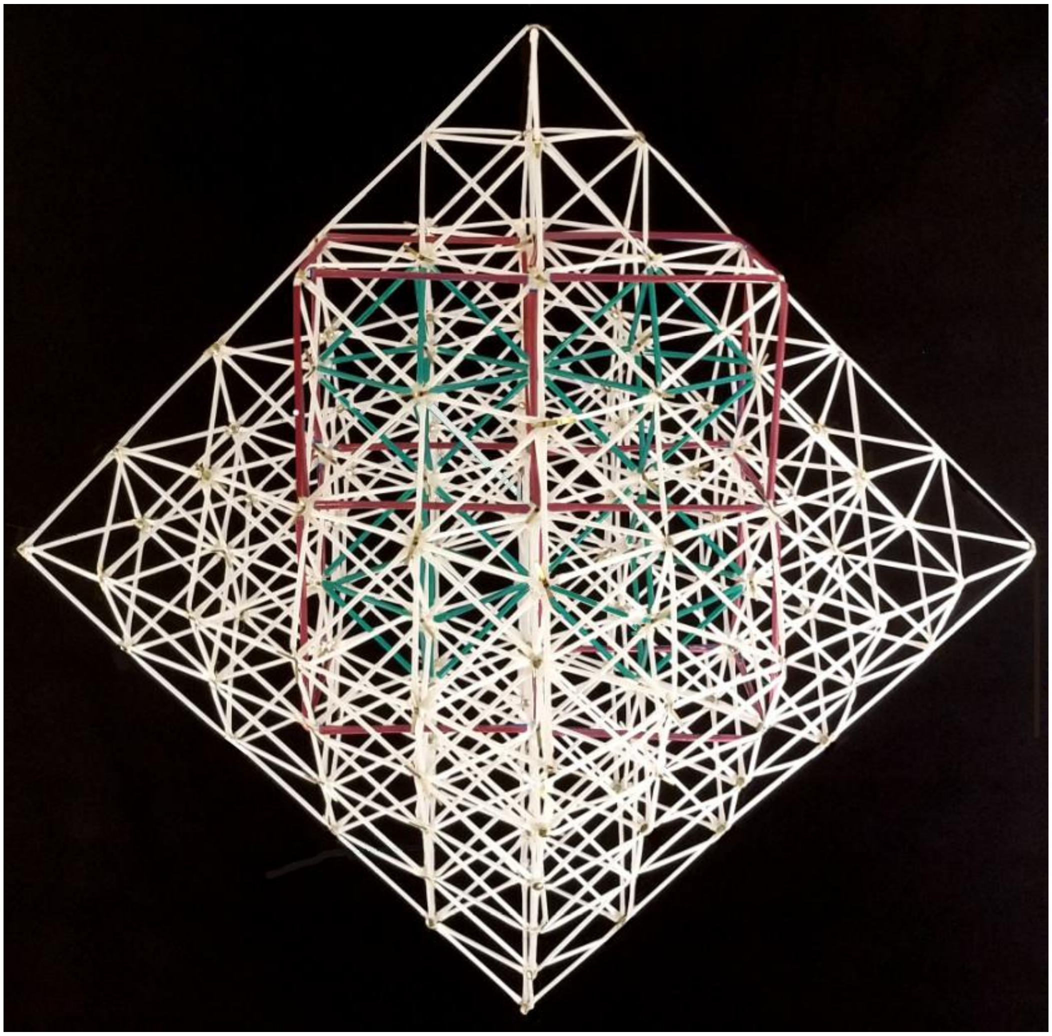


Figure Q6: Tetrahedral-Octahedral Lattice as a stacking of √2 Cubes

For ease of understanding and visualization we can consider our Characteristic √2 Cube as a standalone form, where we see an Octahedron (in green) contained in the Cube (in red) and eight Tetrahedrons (in white) off each face of the Octahedron with a vertex at each corner of the Cube. The reality, however, is that there is no such thing as a standalone form in the context of the 'whole' Lattice. Each 'perceived' standalone form in the Lattice can just as easily be perceived as space between neighboring elements that also appear to be standalone forms, and on and on. The Lattice is a highly ordered network of shared elements in 3D.

It is here that our propensity to reduce things down to characteristic elements, i.e., the particle-mode perspective, butts heads with the notion that all things are interconnected, i.e., the wholistic perspective.

Fragmentation

In his book '*Wholeness and the Implicate Order'* (Bohm 1980), physicist *David Bohm* begins with the consideration of fragmentation and wholeness:

"for fragmentation is now very widespread, not only through society, but also in each individual; and this is leading to a kind of general confusion of the mind, which creates an endless series of problems and interferes with our clarity of perception so seriously as to prevent us from being able to solve most of them".

The central theme of Bohm's thought is the characterization of reality as:

"the unbroken wholeness of the totality of existence as an undivided movement without borders". (Bohm 1980)

Bohm describes our common mode of perception and thinking as one that leads us to understand 'things of nature' as static objects that are decomposed and classified into ever smaller static parts, and then "*reconstructed according to our observations into manageable scenarios."*

Bohm implies that this thinking modality prevents perception of 'unbroken wholeness'. We seem to be biased toward spotting discrete forms, resulting in a fragmented worldview. So, the goal is to develop the ability to sense '*the whole'* in everything we perceive. We return to our Geometric form study to help us understand how fragmentation in perception and thought might happen. Figures Q7 and Q8 provide a great example:

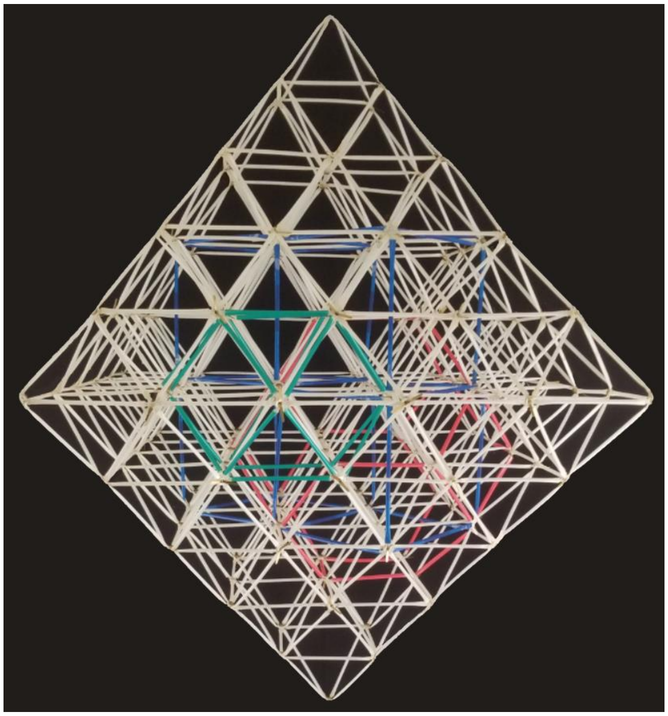


Figure Q7: Cube, Cuboctahedron and Truncated Octahedron Sub-Forms: Edge View

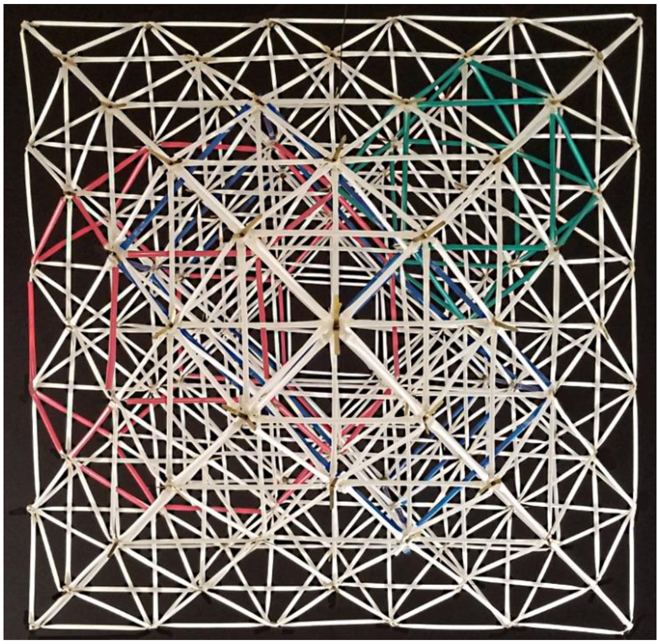


Figure Q8: Cube, Cuboctahedron and Truncated Octahedron Sub-Forms: Vertex View

Highlighted within the Tetrahedral/Octahedral lattice in these figures are the sub-forms of √2 Cubes (in blue), the Cuboctahedron/VE (in green) and the Truncated Octahedron (in red). In classical Geometry these forms are treated as standalone forms. The first a Platonic Solid, the other two are Archimedean Solids. They have been measured and characterized in detail.[16]

Classical Geometry is largely a library of standalone form characterizations built over centuries of study. We have inherited this library along with a mode of perception to comprehend it. When we look at just the highlighted sub-forms in Figures Q7 and Q8 our perception is 'decomposing'. When we see that the highlighted sub-forms are simply aspects of the same space-filling Tetrahedral-Octahedral Lattice, our perception is 'wholistic'. **In the context of the whole, this difference in perception stands in stark contrast.** Once we observe these forms 'in context' we can no longer comprehend the Cuboctahedron 'in isolation' from the Truncated Octahedron, for we now recognize that the former is just a smaller instance of the latter[17]. While they do indeed trace different patterns to the eye, they are really the same thing - the Tetrahedral-Octahedral Lattice - the Triangle-Square Duality. To consider a given sub-form in isolation and then *'reconstruct it into manageable scenarios'* is indeed a bit absurd.

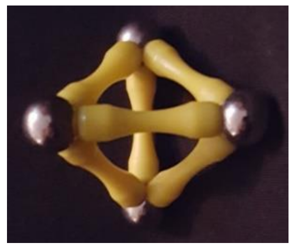
[16] *See Appendix D for more discussion on this topic.*

[17] *Going even deeper into form perception we can also consider the Tetrahedral/Octahedral Lattice as simply the Tetrahedral Lattice. This lattice is essentially a mesh of Tetrahedrons. It just so happens that the space between the mesh of Tetrahedrons are of the form of Octahedrons. Again, a shift of perception is required.*

This small example should make us consider the inherited assumptions and reflexive perspectives we bring into our thinking in general, and how this unwittingly pushes us into fragmented, particle-mode, thinking.

Inherited Assumptions

As a simple practice, a fundamental shift in our perspective can be experienced by re-comprehending the Tetrahedron. By and large, we have inherited the classical isometric perspective, shown below on the left, where we likely see the Tetrahedron as an opaque static object, that ‘pointy three-sided pyramid’ – end of story.



How much richer the perspective of the Tetrahedron on the right that shows the ‘dynamic’ of point equilibrium at play - the spatial harmonization of both distance and direction. The curious truth-seeker, I argue, will gain a more accurate perspective about natural and man-made forms more easily by building node-based models than by observing solid models. Yet, historically, we find that students of Geometry predominantly worked in the medium of solid models. This is probably because resilient strut material and strong attachment mechanisms for stick-and-hub (node) modeling were simply not available to them. It is likely that stick-and-hub models were patiently constructed in the past but, absent durable material, the models would have been too fragile to survive handling and storage over long periods of time. By contrast, solid models could be carved out of the readily available, pliable, and durable material of clay, using simple tools. Such models would last for generations and serve as a staple teaching tool for passing down knowledge. Thus, solid models and the observational perspective they enforce, has dominated human thought.

As mentioned, ‘solid Polyhedra’ have been thoroughly characterized over a long historical timeframe. As current recipients of this body of knowledge, we must frankly acknowledge though that Geometry study has become arcane and put out of the reach of most people. Although everyone would benefit from an intuitive understanding of the provenance of Geometric pattern, both in Natural and Man-made form, there is a cultural failure to deliver it. I argue that qualitative spatial literacy must be considered a birthright, so let’s dig deeper into this failure:

It bears restating that Geometry is a language of its own. It is easy to mistake the symbolic language that characterizes Geometric form, for the spatial language of Geometric form itself. The former is an abstract second-hand comprehension, the latter is a direct intuitive connection to Nature itself.

While paying due respect to the field of Solid Geometry, a critique is also in order:

Classical Geometry rests upon the premise that base forms are ‘regular’. In the term ‘regular’ we should recognize, once again, the premise of ‘equality’ of parts, just as we posited earlier by the premise of the equality of strength of our imagined points in space. In classical Geometry the regular base forms are the Platonic solids, e.g., Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron. New forms are derived by executing the same action on all same aspects of the base form, for example, truncating/shaving all edges, truncating all vertices, swapping corners for faces and vice versa to find ‘duals’, or stellateing all facets to equal height. It should come as no surprise then that the resulting forms are also ‘regular’. Subsequent actions on the derived forms will again result in new regular (or uniform) derived forms, and on and on.

To restate simply: A given regular solid, subjected to a consistent transforming action (transform), produces a new regular solid, i.e., (Regular IN → Same Transform on all Like Aspects → Regular OUT). This is the underlying workflow of classical solids study. (*Like sculpting clay, or carving ice, anyone can do it! There is no impenetrable mystery here.*)

Now, given infinite possible degrees to which a transform can be applied, (e.g., the depth of the truncation, the height of a stellation, etc.) it should also be appreciated that <u>an infinite number of derivable forms are possible</u>. If we shave enough, we work our way towards a sphere-like polyhedron where the number of facets approaches infinity. In practice, however, only mathematically special forms in that infinite set are significant and passed down (the Archimedean Solids is one example).

In this way, over the long arc of history, new forms were derived, characterized by measurements of their features (e.g., counts of vertices, edges, facets, angles, etc.) and named per a structured nomenclature. A body of knowledge was thus built, and as previously mentioned, most people encountering it are perplexed by it: What do all these measurements mean? What are they used for? Who uses them and why? Most are intimidated and turn away, a tiny few pursue it in university-level studies.

Ultimately, what is missing in the current art of Geometry study can be illustrated by a comparison of a still photograph of a person to a video of the person interacting with others. The video ultimately gives the viewer a better ‘feel’ for the subject’s personality. Even if we measured all the various aspects of the person’s physical form and published those values next to the photograph, to most people, the video of the ‘person in motion’ will still be a more valuable characterization. So too, simply observing a node-based Geometric model in a progressive buildout, will be for most people, more valuable than observing a fully quantified static model. Unfortunately, such qualitative study is largely absent our classrooms and in our culture of play.

Epilogue

Our journey with Geometry modeling has served us well. It extended the reach of our perception allowing us to sense operating principles behind Nature's form-making that we would not able be to visualize in our imagination. It gave us a basis to understand how, with a simple palette of Planes, Lattices, Cages and Tubes, Nature has tremendous freedom and power of variation to construct the diverse physical world of form that we see.

From this journey, we may conclude that the true significance of Geometry modeling lies not in a plethora of measurements and mathematical characterizations, but in its ability to give our imaginations a **'qualitative feel'** for how, in Nature, complex physical structure can be built from simple patterns. Such an impression is transformative.

Appendix A: Whole Number Dilemma

We wish for a simple elegance in all of Nature's forms, where all form is based solely upon patterns of whole numbers. Pythagoras purportedly exclaimed "All is Number!". However, we find that we must deal with irrational numbers, such as the $\sqrt{2}$ diagonal of the square, the $\sqrt{3}$ diagonal of the cube, and other non-integer node relationships.

Recall that the Icosahedron represents a cage pattern of twelve nodes that are in unit-length equilibrium with each other and perfectly balanced around a center point. At first blush it appears that the twenty equilateral faces of the Icosahedron might be assembled from twenty Tetrahedrons, each with one vertex meeting at the center. While close, this does not work. Let's analyze:

Assume, for argument sake, that the Icosahedron <u>can</u> be derived from twenty equilateral Tetrahedrons mated at their faces and all touching at a single vertex at the center. In Figure G1 we isolate one Tetrahedron in this scenario, shown with three face edges in blue and three interior edges in red. The struts in red go inward and intersect at the center of the form. The struts in blue form one face of the Icosahedron.

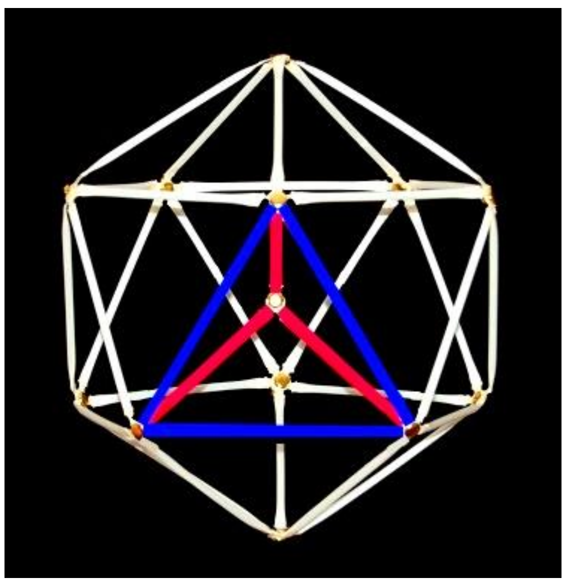


Figure G1: Icosahedron with False Interior Tetrahedron

We now disprove this assumption by showing that the inner struts in red are not equal in length to the unit-length blue struts, and therefore are not equilateral Tetrahedrons, and thus, equilateral Tetrahedrons do not comprise an Icosahedron.

If Tetrahedrons can indeed create an Icosahedron, then the sub-form of Figure G2 is possible.[18] It is comprised of five unit-length equilateral Tetrahedrons. There would be twelve such sub-forms inter-meshed to form the full Icosahedron, somewhat like in Figure D5.

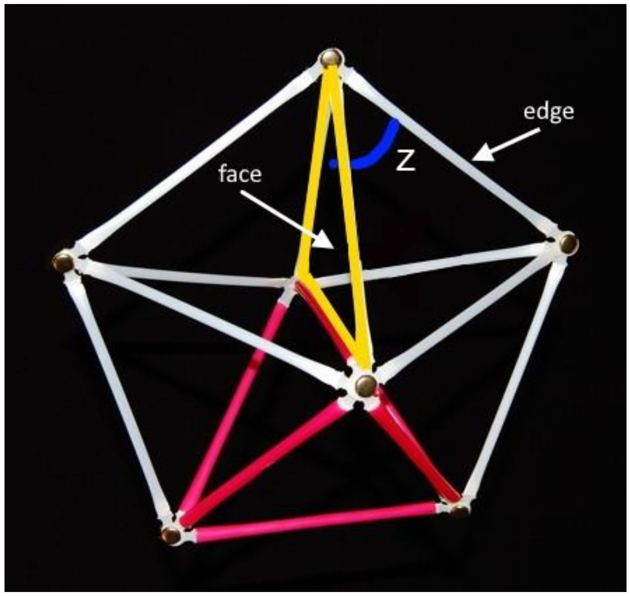


Figure G2: False Icosahedron Sub-Form

We note that the outer edges of Figure G2 form a Pentagon. A Pentagon has interior angles of 108º. Each interior face (one is shown in yellow) bisects this angle, therefore angle Z = 54º. However, through a trigonometric analysis of the regular Tetrahedron we find that the angle between an edge and an opposing face, i.e., angle Z, is 54.74º. A small difference, but nevertheless, we have a contradiction thus proving that the sub-form in Figure G2, hence our assumption that the Icosahedron can be created from twenty regular Tetrahedrons, is false.

If we view the Icosahedron from a 'Solids' perspective, we could determine twenty non-regular tetrahedrons that would fit together to create the twenty faceted Icosahedron. The tetrahedrons would be slightly squashed with one equilateral Triangle face and three isosceles faces. But we want to view the Icosahedron as a 'force equilibrium'. The above analysis reinforces our observation of the Icosahedron, and its expansion in a sequence of concentric Cages, as having a

[18] *The modeling tool used for the model in Figure G2 is a 'compensating' modeling tool. It has flexibility to compensate strut length to a small degree and thus can compress the interior struts to slightly less than unit-length. It is thus able to form slightly squashed Tetrahedrons and Pyramids. The beauty of the compensating modeling tool is that it allows us to find the pattern of subsequent Icosahedral cages building out from smaller cages. In build-out, pre-imagining the pattern of the next larger cage is exceedingly difficult. It is much easier to do when you can experiment with different possible patterns using the modeling tool itself.*

spatial center but not a point force at center, for it is simply not possible for a center point to spatially equalize with the twelve equalized points that surround it. The **only** pattern of twelve points at equilibrium around a center point is the pattern of the VE (Cuboctahedron) shown in Figures C5 and C8.

Appendix B: The Splendor of Self-Similarity

Figure H1 is a mélange of Icosahedral buildouts that illustrate the notion of Self-Similarity with the patterns of five occurring at the different layers of the models.

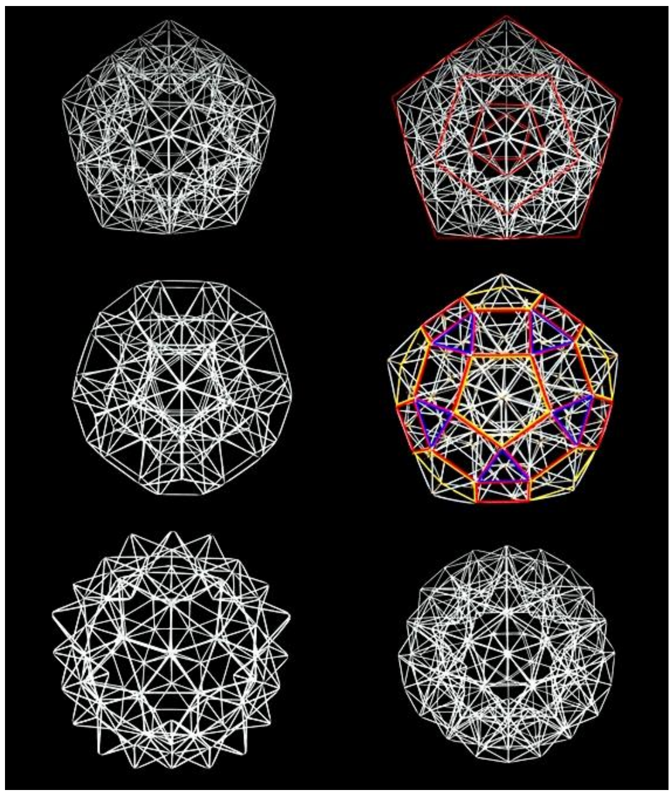


Figure H1: Icosahedral Buildout Variations

Figure H2 is a meditation on Icosahedral Self-Similarity:

Figure H2: “As Above – So Below”

Appendix C: The Number Seven

In the field of Sacred Geometry, the number Seven (7) is enigmatic. It is not formable by the compass-and-straightedge techniques of Sacred Geometry* and does not show up in classical study of solids. However, when viewing nodes of a 3D cubic pattern the significance of the number seven is evident as seen in this isometric view of a constellation of seven orthogonal points at unit length distance.

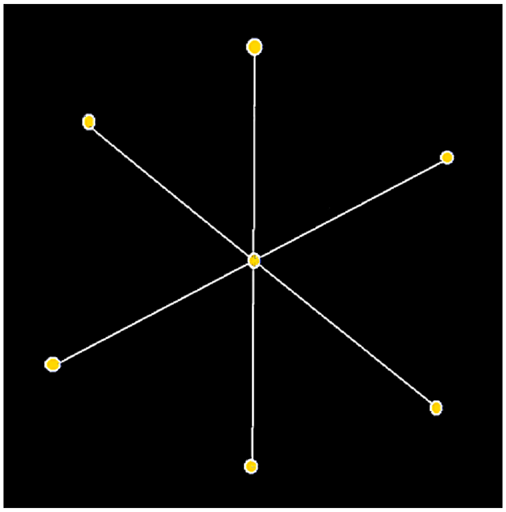


Figure I1: Perfect Spatial Balance

This is the simplest expression of point equilibrium, an equilibrium with two aspects, the equalization of **extent** (unit length) and **direction** (orthogonality).

* Notice the symbol of the Masons below - Compass and Square juxtaposed. These are the tools of Sacred Geometry but should also symbolize for us the notion of the spatial balance of extent and direction.

Appendix D: Expansion of the Tetrahedral-Octahedral Lattice

In the discussion on Fragmentation above we saw that the Cuboctahedron (Vector Equilibrium) and the Truncated Octahedron are but 'aspects' of the Tetrahedral-Octahedral Lattice yet they are commonly studied as standalone Archimedean forms. In buildout, as the node count increases[19] even more unique aspects of the Lattice are possible. Figures J1 and J2 show another variation of the Cuboctahedron sub-form.

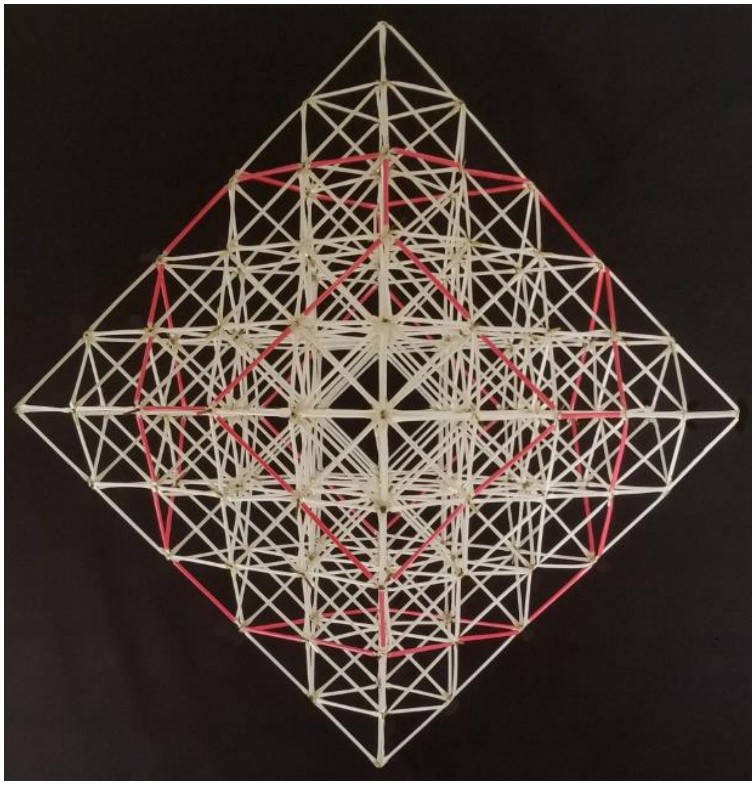


Figure J1: Truncating a Large Tetrahedral-Octahedral Lattice

In Figure J1 each corner of the Tetrahedral-Octahedron Lattice is about to be truncated per the red lines to produce six Square facets of 2x2 unit squares. Recall that with the Archimedean Cuboctahedron unit Squares facets are complemented by Triangle facets. In the Archimedean Truncated Octahedron, unit Square facets are complemented by Hexagon facets. Figure J2 shows how the 2x2 unit Square facets are complemented by what might be termed 'hybrid' Hexagons. Instead of regular Triangle or Hexagon faces, we have here non-regular Hexagons where three sides are 1-unit-length and three sides are 2-unit-length.

[19] *The terms 'node count' and 'frequency' are virtually synonymous. The more nodes, the higher the frequency of the form.*

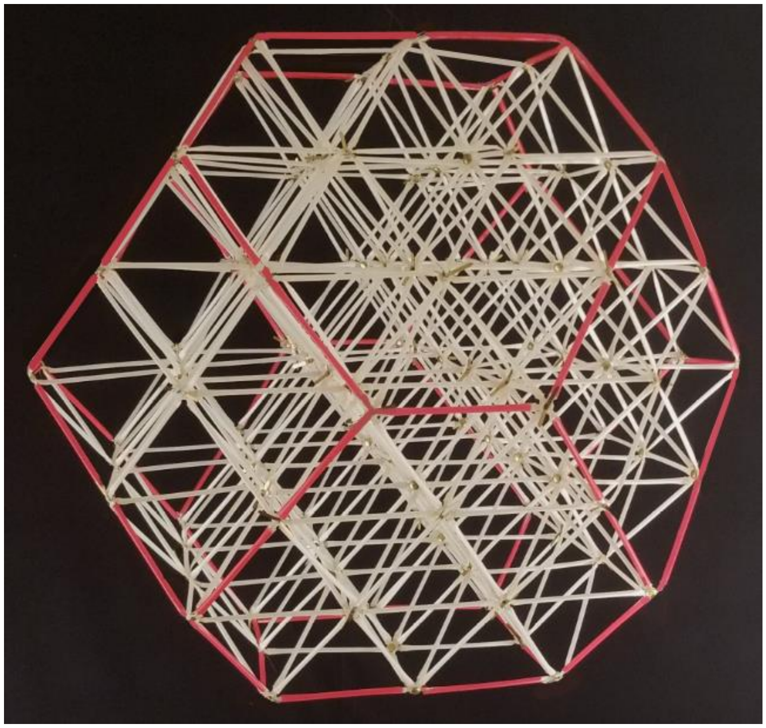


Figure J2: Form of the Second Expansion of Cuboctahedron

Figure J2 basically shows us what the base pattern of the Cuboctahedron/VE looks like in expansion. Where the Truncated-Octahedron is the first expansion pattern from the Cuboctahedron, Figure J2 shows the second expansion pattern. However, unlike the Cuboctahedron and the Truncated Octahedron, the face pattern of Figure J2 is not among the Archimedean face patterns.

As the Tetrahedral-Octahedral Lattice grows larger there are likely more unique face patterns possible. What Triangle/Hexagon faces are possible when the Square facets are 3x3 unit Squares, 4x4, etc.?